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An analysis of the Fusée Céramic System and the design of form-active environmental friendly roofs

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An analysis of the Fusée Céramic System and the design of form-active environmental friendly roofs

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Contents

Summary

Just after World-War II building materials were scarce, architects and engineers had to design buildings using not much cement and steel. In French an architect, Jacques Couëlle, had invented a system with céramique infill elements to reduce for structures of concrete the self-weight and need of cement and steel. In the fifties and sixties of the twentieth century these céramique elements, known as Fusée Céramique elements, were used widely in France, North Africa and the Netherlands, mostly for barrel vaults and shells. Nowadays most of these structures are pulled down and the remaining buildings do not meet the demands of the present concerning climate comfort, insulation and safety. This thesis analyses the structural design of cylindrical Fusée Céramique roofs in the context of those days. The effect of the céramique infill elements for the time dependent deformations, stiffness and load bearing capacity, including second order, is studied. To save the few remaining buildings for the coming generations the possibilities to strengthen these structures with slender light elements of steel are explored. The effect of the strengthening is described for a Fusée Céramique vault, designed and constructed in the past.

Reinforced concrete is a widely used building material with many advantages. Unfortunately the production of both reinforcement and cement is quite energy intensive and causes the emission of greenhouse gasses as NO_2, NO and CO_2. Reducing the need of cement is a relative simple way to reduce the emission of these greenhouse gasses.

In practice roofs are seldom really flat but curved or at least slightly inclined, to drain rainwater and snow. Structurally curved structures, transferring loads as a surface-active or form-active structural system, are very efficient. The need of material and the self-weight is pretty low. This can be very useful if in the future the potentials of roofs for producing food and energy are used more often and these roofs must be designed for much heavier payloads as usual at the present.

For form-active and surface-active roofs of concrete the self-weight and need of steel and cement can be reduced further with light infill elements. This study of the Fusée Céramique system shows that infill elements can save cement, self-weight and reduce the environmental load. The possibilities to save cement and reduce the environmental load with infill elements are studied with respect to the construction techniques of the present.

The design of prefabricated cylindrical vaults, composed of segments following a part of a circle and strengthened with slender ties of steel to reduce the bending stresses, is described. To produce prefabricated cylindrical barrel vaults efficiently a positioning of tubes perpendicular to the span is preferable. The effect of this infill concerning the load transfer is analysed. Models of a prefabricated element, with tubes positioned perpendicular to the span, are tested to define the structural bearing capacity of prefabricated barrel vaults.

Samenvatting

Vlak na de tweede wereldoorlog waren de bouwmaterialen schaars, architecten en ingenieurs moesten gebouwen ontwerpen met een beperkt gebruik van cement en staal. In Frankrijk had een architect, Jacques Couëlle , een systeem bedacht om met keramische invulelementen het eigen gewicht van betonconstructies te reduceren en te besparen op cement en staal. Deze keramische elementen, bekend als Fusée Céramique elementen werden in de vijftiger en zestiger jaren van de twintigste eeuw op een ruime schaal toegepast in Frankrijk, Noord-Afrika en Nederland voor hoofdzakelijk cilinderschaaldaken en koepels. Momenteel zijn de meeste van deze constructies gesloopt en de resterende gebouwen voldaan vaak niet meer aan de hedendaagse eisen voor het binnenklimaat, de isolatie en de veiligheid.

In dit proefschrift wordt het ontwerp van de Fusée Céramique daken geanalyseerd in de context van die tijd. Het effect van de keramische invulelementen op de tijdsafhankelijke vervormingen, de stijfheid en het draagvermogen, inclusief het tweede orde effect, wordt bestudeerd. Om de weinige nog resterende gebouwen te bewaren voor het nageslacht wordt onderzocht hoe deze constructies versterkt en verstijfd kunnen worden met slanke en lichte elementen van staal. Het effect van het versterken wordt beschreven voor een in het verleden ontworpen en gebouwd Fusée Céramique schaaldak.

Gewapend beton is een veelvuldig toegepast bouwmateriaal met vele voordelen. Helaas vergt de productie van cement en staal veel energie en komen bij de productie gassen vrij als NO, NO_2 en CO_2 , die bijdragen aan het broeikaseffect. De uitstoot van deze broeikasgassen kan op een eenvoudige wijze worden gereduceerd door de hoeveelheid cement in de betonconstructies te verminderen.

In de praktijk zijn daken, voor het afvoeren van regenwater en sneeuw, vrijwel nooit geheel vlak , maar gekromd of op zijn minst enigszins hellend. Constructief gezien zijn gekromde dakconstructies, die de belastingen als oppervlak-actieve constructie of als vorm-actieve constructie kunnen afvoeren, zeer doeltreffend. Voor deze constructies is zowel het materiaalgebruik als het eigengewicht tamelijk gering. Dit is zeer nuttig als in de toekomst de mogelijkheden van daken om energie en voedsel te produceren vaker benut gaan worden en deze daken ontworpen moeten worden op een hogere nuttige belasting dan momenteel gangbaar.

Voor de betonnen vorm-actieve en oppervlak-actieve constructies kan het eigengewicht en de benodigde hoeveelheid cement en staal verder worden beperkt met lichte invulelementen.

Dit onderzoek naar het Fusée Céramique systeem toont aan dat met ingestorte elementen minder cement nodig is en tevens het eigengewicht en de belasting op het milieu verminderd wordt. Onderzocht wordt hoe met invulelementen, uitgaande van de momenteel gangbare uitvoeringsmethoden, het cementgehalte en de milieubelasting teruggebracht kan worden..

Het ontwerp van geprefabriceerde cilindrische schaaldaken met een cirkelvormige kromming wordt beschreven, die om de buigspanningen te reduceren zijn versterkt met slanke stalen staven. Om de geprefabriceerde schaalconstructies efficiënt te kunnen maken, worden de in te storten buizen bijvoorkeur niet evenwijdig maar loodrecht op de kromming geplaatst. Het effect van deze plaatsing op de krachtsafdracht wordt geanalyseerd. Om het draagvermogen van deze geprefabriceerde cilindrische schaalconstructies te bepalen zijn modellen van een geprefabriceerde schaalelement met sparingbuizen dwars op de overspanning beproefd.

Notation

A	cross section;
E	Young's modulus;
F	Force;
H	thrust;
I	second moment of the area;
M	bending moment;
N	normal force;
N_{cr}	buckling force;
P	post-tensioning force;
V	vertical reaction;
W	moment of resistance;
a	half span;
e	eccentricity;
f	rise of the vault;
f_x	ultimate stress;
h	height of a section;
k	reduction factor for the creep;
l	span of the barrel vault;
l_c	buckling length;
n_{cr}	buckling ratio;
n_f	ratio stiffness fusées to stiffness concrete;
n_s	ratio stiffness rebars to stiffness concrete;
q	equally distributed load;
q_e	equally distributed live load;
q_g	equally distributed permanent load;
s	length of the curve of an arch between the top and support;
t	depth of the vault;
u	deformation;
w	width;
x	coordinate X-axis;
y	coordinate Y axis;
α	angle; factor;
β	angle; factor;
ϕ	angle; creep factor;
σ	stress;
ε	specific deformation;
ε_{rc}	specific deformation of concrete due to shrinkage;
ψ	factor reduction buckling length.

Introduction

Fusée Céramique System

Just after World-War II infills were used often to reduce the cost of structures of concrete. For floors tubes of cardboard and for vaults tubes of céramique, the so-called Fusée Céramique tubes, were used to reduce the need of cement and save cost. Nowadays the Fusée Céramique system is almost forgotten, but half a century ago this system was well known and competitive for roofs above industrial buildings. A fusée is a cylindrical céramique tube, which was embedded in concrete walls and roofs, mostly barrel vaults and domes, to reduce weight and cement. Just before the second Word War, a French architect, Jacques Couëlle, invented this system. In World War II many buildings were destroyed, so after the war the need for buildings was huge and the building industry was booming. Consequently the cost of materials was rising and architects and engineers had to save cement, steel and other building materials. For structures of concrete the self-weight as well as the need of cement and steel could be reduced considerately by embedding a light cost effective infill. Of course not every infill is suitable for structures of concrete, the infill must be chosen carefully. Combining materials with varying features can affect the load bearing potential. Due to shrinkage and thermal expansion composites can be subjected to internal stresses potentially causing cracks and a reduction of the load bearing capacity. Possibly for some vaults built with Fusée Céramique elements the safety is not fulfilling the demands of the present and need to be strengthened.

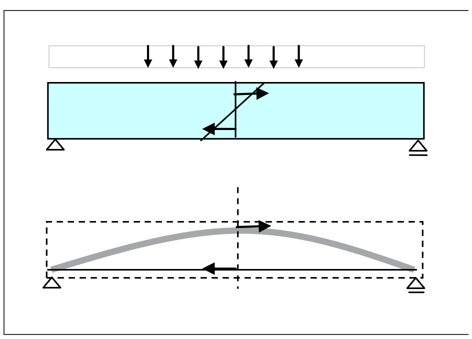
Saving building materials

At the moment many people are concerned about the environment of our planet. The number of people living on this planet is rising exponentially. To feed these people increasingly woods and wastelands are cultivated and transformed into urban areas at the cost of biodiversity. Pollution threatens the delicate eco-systems in the oceans and jungles. Burning fossil energy pollutes the atmosphere. Due to the greenhouse effect the average temperature has been rising for more than a decade. The melting of glaciers and polar ice will lead to rises of the sea and ocean level, possibly coastal areas will be flooded. In the sixties of the XXth century scientists warned us for the consequences of overpopulation and the use of non-sustainable energy for the environment. Many people are aware of these problems and are concerned to stop the climate change. Consequently the consumption of fossil energy must be reduced, for example by saving energy and transferring from fossil to renewable energy. The building industry can contribute much to save energy. An important part of the yearly energy consumption is needed to create a comfortable climate in buildings. Probably this will change in the coming decades. To save energy architects design buildings that do not use fossil energy for heating or cooling. Consequently the next step in reducing the total energy consumption will be the reduction of the environmental load of the materials. Professor Haas expects: ''that with a greening of energy in the next fifteen to twenty years, the material share concerning the environmental load of buildings will increase to 48%. Materials are our next problem after energy'' [Was09]. This will change the building industry significantly. Just as in the past architects and engineers will optimise buildings to reduce the need of the energy for the production, transport and assemblage of building elements. The better part of the materials is needed to make the load-bearing structure of a building. Consequently just as in the past the reduction of the self-weight of the structure will be an essential part of building design.

Curved roofs

Curved roofs designed as a form-active structure system can transfer the loads most efficiently. Engel defines these systems as: "Form-active structure systems are structure systems of flexible non-regid matter, in which the redirection of forces is effected through particular form design and characteristic form stabilization" [Eng99]. In practice most building structures are section-active structures, even for roofs. Engel defines these systems as: "Section-active structure systems are structure systems of solid rigid linear elements - including their compacted form as slab - in which the redirection of forces is effected through mobilization of sectional forces" [Eng99]. To drain the rain water roofs need a curvature or slope. For flat roofs ponding is a real threat, every year some flat roofs collapse due to the accumulation of rainwater. A roof can be designed also as a surface-active structure system, but building surface-active structures is complicated and labour intensive, consequently shells are not often built nowadays. Otherwise new techniques of design and construction are being developed. At the beginning of this century free forms became quite popular. Thanks to the CAD-CAM technology these complex forms could be designed and built in spite of the irregular form. Single curved surfaces are easier to make than double curved surfaces and the cost of construction will be less. Selecting materials that use less fossil energy, such as organic materials, can thus reduce the environmental load of structures. Unfortunately organic materials are mostly not very stiff and strong, so the dimensions are substantial. Consequently the volume of these buildings will be increased. The environmental load can be reduced too by developing new high-tech materials. Generally the environmental impact of light building materials or lightweight elements is less than the environmental impact of heavy materials such as concrete and masonry. Nowadays structures for low-rise buildings, housing for example workshops, swimming pools, sporting halls and shopping malls, are mostly composed of light materials. Especially for large spans light materials or elements are preferred, so the self-weight is not consuming most of the load bearing capacity. But the capacity of these light constructions to accumulate heat or cold is poor. Passive energy is ideal for low temperature heating and cooling systems, using green energy. During the winter the heat is accumulated into the floors and walls at the night and during the day the heat is returned. In the summer the process is used to cool the inner spaces during the day by restoring the heat lost from the core elements during the night. Embedding infill elements can reduce the environmental impact of concrete structures. Concrete is composed of a mixture of cement, sand and gravel. The environmental impact of sand and gravel is small. The environmental impact of cement can be reduced considerately, for example by using blast furnace cement and fly ash. Blast furnace cement is a by-product of blast furnaces producing steel. Fly ash (pulverized fuel ash) is a by-product of energy plants. The use of fossil energy for the production of cement can be reduced too by using organic waste for the furnace of the plant. Reduction of the self-weight will reduce the need for steel reinforcement. The self-weight can be reduced with light aggregates and infill elements. In the past the self-weight of floors was reduced with tubes of cardboard, at the present the self-weight is reduced with spherical bals and polystyrene boxes.

Actually both strategies, minimizing the environmental load by form design and reducing the embodied energy/self-weight were applied 70 years ago already. In France the architect Jaques Couëlle invented a system known as Fusée Céramique. Just after World War II curved concrete roofs composed with ceramic tubes, known as the Fusées Céramique, were built in France and The Netherlands. Fifty years ago most buildings were constructed with concrete roofs. For large spans the self-weight was compensated by form-design. Folded plates, cylindrical vaults and double curved shells can span considerable lengths with a minimal thickness and a minimal use of material. Thanks to the reduction of the self-weight the Fusée Céramique roofs were competitive to roofs with a steel or timber structure. In the sixties the costs of the labour were rising faster than the costs of materials, so these vaults, constructed in situ, became less competitive and were thus not built any more. Nowadays the cost of the environmental impact has to be included and this will change the building industry again and force architects and engineers to find new solutions. Developing new materials is expensive. A lot of research is needed to demonstrate that new materials can be applied safely. Learning from the past is cost effective. Buildings standing for fifty years show problems and time dependant effects distinctly. Nevertheless the Fusée Céramique roofs performed well for more than fifty years. Nowadays prefabricated flat hollow core plates can be applied for spans up to 18 m, the ratio of thickness to span is only $^1/_{40}$. To pre-tension these elements, high tensile steel wires are pre-tensioned on a frame and the concrete is extruded. When the concrete has set the long plates are cut to the required length. Due to this process of production these elements are flat slabs and cannot be used for curved roofs. With curved hollow core elements the maximal span of curved roofs can be increased and the ratio of thickness to span can be reduced considerately. For example the ratio of thickness to span of a fusée vault was often less than $^1/_{150}$. These structures were well designed according to the features. Structurally concrete can resist compressive stresses very well but the resistance to tensile stresses is poor. Usually, to resist tensile stresses, structures of concrete are reinforced, commonly with steel. If a reinforced concrete beam is subjected to bending then a part of the section resists the compressive stresses and the reinforcement resists the tensile stresses. The concrete in the tensioned part of a section in bending is cracked and not transferring loads. Structurally it is efficient to remove the parts between the tensile and compressive zone that only transfer limited shear stresses. The following figure shows a beam subjected to a lateral load. If the tensioned and not highly stressed zones are removed, the load is transferred by a slender compressed arch and tensioned tie. This reduces the self-weight considerably. Generally form-active structures can be designed much thinner than section- active structures. For example beams are designed with a ratio height to span of about $^1/_{10}$ and arches are designed with a ratio rise to span of $^1/_{40}$ at minimum. Thus structures of concrete are preferentially form-active to reduce material consumption and to minimise dead weight and embodied energy. Unfortunately in practice it is not always possible to apply arches. For floors the upper surface has to be flat, so most floors are section-active structures.



Improving the efficiency of a beam of concrete by removing material and changing the beam into a low-rise arch.

Arches and vaults

This thesis focuses on low-rise barrel vaults. The effect of the end walls of a barrel vault will be small if the length of the vault is more than three times the span, then these vaults and vaults with no stiffening at the ends can be schemed as an arch. In practice the Fusée Céramique vaults were partitioned by dilatations with a centre-to-centre distance of about 5 m. Due to the partitioning these structures can be schematised as arches. Nevertheless for the design and load transfer the following differences must be considered. Generally the ratio thickness to span is for a concrete arch $^1/_{40}$. This is much more than for a low-rise barrel vault with a ratio thickness to span $^1/_{150}$. For these slender low-rise vaults buckling can be a serious threat. Generally the loads acting on arches are much more than the loads acting on vaults, also the normal stresses will be higher than for vaults. For the Fusée Céramique vaults described in this thesis the normal stress was seldom more than 1.0 MPa. For arches the ratio rise to span is seldom smaller than $^1/_5$. Due to the modest stresses the ratio rise to span can be for low-rise vaults much smaller than for arches. The Fusée Céramique vaults, described in this thesis, were designed with a ratio rise to span of $^1/_8$. Consequently the design rules developed for these low-rise vaults cannot be applied for arches.

Infills

The technique of construction resembles the techniques applied by Eladio Diëste for shells of reinforced masonry. Comparing these techniques will show the context of the fusées at that time. Thanks to the ceramic infills, the need of cement and the self-weight of the vaults were reduced considerately. Consequently the Fusée Céramique barrel roofs were competitive for roofs with a span up to about 24 m. Later other infill elements were introduced to reduce the weight of concrete floors and to save cement, for example cardboard tubes, cassettes, spherical balls (bubble deck), boxes of polystyrene and so on. Generally combining varying materials causes complications, often with adverse consequences, especially when the physical features vary. Due to the shrinkage of the concrete the composite is subjected to internal stresses too. Possibly the concrete, enveloping the fusée elements, is cracked due to these stresses; consequently the stiffness of the vaults and the safety concerning buckling is decreased. Nevertheless some of these buildings are still in use. The effect of the time dependent deformations for the load transfer and safety will be analysed in this thesis.

Focus

This thesis focuses on the load transfer for form active structures of concrete made with light infill elements to reduce the self-weight, need of cement and environmental load. To learn from the past the load transfer is studied for Fusée Céramique vaults. The schemes, theories, idealizations and assumptions, practised during the nineteen fifties are analysed, validated and discussed. For a remarkable cylindrical vault, constructed in Woerden, the design is reconstructed and the structural bearing capacity is defined. Unfortunately most of these buildings, just as the building in Woerden, are pulled down already. To remain some of these structures and use these buildings safely the possibilities to strengthen these vaults are explored. Currently the worldwide attention for the environment should stimulate architects and engineers to seek sustainable, durable and environment-friendly techniques. These will save energy and reduce CO_2 emission. To reduce the footprint of structures of concrete, especially vaults, further the possibilities of infills are studied with respect to the construction techniques of the present. To facilitate the construction of prefabricated circular vaults the weight and need of concrete is reduced with cardboard tubes positioned perpendicular to the span. The effect of the direction of the tubes for the load transfer is studied. Tests are made for circular prefabricated cilindrical vaults with cardboard tubes positioned perpendicular to the span to validate the concept and to create light roofs able to resist the loads for public activities or the production of food and energy. Thus possibly the Fusée Céramique concept can be transformed into new technical systems and is this study a starting point to save material and to decrease the environment load for cilindrical vaults, to create sustainable green roofs contributing to a human friendly climate in urban areas.

1 The Fusée Céramique System, history, construction, dimensions and context

This chapter describes the Fusée Céramique system. In successive paragraphs the history, inherent advantages, the technique of construction, the form and the dimensions are described. The technique of construction resembles the techniques applied by Eladio Diëste for shells composed of reinforced masonry and the techniques, still practised in India, to built the Guna vaults. Comparing these techniques will show the context of the fusées at that time.

§ 1.1 A short description of the history of the Fusée Céramique System

Halfway the XXth century the building industry was booming, consequently the costs of scarce materials were rising and architects, engineers and contractors had to find alternatives to save cement and steel. During Word War II the French architect Jacques Couëlle had invented a system to build structures using cylindrical ceramic elements embedded in concrete. Actually Byzantine and Roman engineers applied before hollow ceramic elements. For example ceramic elements were built in, fifteen centuries before, in the church of San Vitale in Ravenna [Eck54]. The slender branches of bamboo, composed of tubes connected and stiffened by nodes, inspired Couëlle to design the Fusées Céramique System. The ceramic tubes have a conical top to join the elements. This enables the conical top to be pushed into the rear of the next tube. The fusées were used as infill for concrete walls, floors and roofs. Thanks to the conical top the elements can be rotated slightly at the nodes, so these elements can easily be used in curved roofs.

FIGURE 1.1 Fusee Céramique element saved from building Q in Woerden. Photo taken by author.

During World War II the army recognized the advantages of the Fusées Céramique elements and used this system for bridges, barracks and shelters. A large factory was built in Marseille, which after the war had a lot of the ready to use fusées in stock. Many people had lost their homes and the architects André Bruyère and Fernand Pouïllion saw possibilities to use these elements for temporary housing. In Marseille a large resort, the Arenas Camp, was built for emigrants from North Africa [Dub00]. Beside in France the system was introduced in North Africa, for example Morocco [Lan49], too. The improved heat insulation, due to the hollow core elements, was appreciated in North Africa, lowering the indoor temperatures during the summer. Further the French army used this system for defence buildings in North Africa [7]. A few years later the system was introduced in Belgium and the Netherlands. A factory, the 'N.V. Nederlandse Fusée Céramique Maatschappij Nefumij', was erected in Echt, which could produce 10 million Fusée elements a year [Lan49]. The engineering and technique of building were described in journals [Toe53] and technical books [Vri55] and many barrel vaults were made, mostly for industrial buildings such as warehouses, factories and garages. Occasionally this system was used for prestigious buildings as churches and railway entrance buildings. The elements were also used for domes. For example in France the church Saint-Jean Bosco in Biollay was roofed with a dome, designed by the architect Pierre Jomain. In the Netherlands the architect H. van Wissen designed a dome for the st.Raphael-Exodus church in Hengelo [11]. The architect H.G.J. Schelling designed a dome for the entrance of the Railway station in Arnhem. This dome, with a diameter of 8.77 m, was built in 1954 [Roo09] and pulled down. A workshop of the Pastoe factory was roofed with conoid shells to enlighten the interior with roof lights, see figure 1.2. During the sixties the cost of labour was rising significantly and this system could not compete anymore with other systems. Nowadays this system seems almost forgotten.

FIGURE 1.2 Conoid shells roofing the Pastoe workshop in Utrecht. Photo: Katja Effting fotografie

Advantages/disadvantages

During the introduction the benefits of this system were emphasized. Langejan [Lan49] described the following advantages:

- a reduction of the self weight of 25% - 40%;
- a saving of cement up to 50% -70%;
- an increase in thermal insulation of 30%-40%;
- uncomplicated construction;
- formwork can be stripped earlier.

Further Van Eck [Eck54] mentioned that the acoustic resistance was practically identical to the acoustic resistance of massive structures.

During the fifties of the past century the building industry was booming and consequently steel and cement became difficult to obtain, thus limiting the use of cement was important. For floors many systems, as for example Nehobo, Steno, Stalton, Omnia, Kwaaital, Cushfeller and Dato floors, were developed to reduce the self-weight and the cement content. The Nehobo, Steno and Stalton floors were composed of ceramic elements. The Stalton floor was composed of prestressed ceramic beams and ceramic infill elements. This floor did not need a formwork. The Steno and Nehobo floors were composed of hollow ceramic elements. The reinforcement was positioned into the voids between the elements.The voids and top layer were filled with cast concrete. These floors did not need a casing, but the ceramic elements had to be supported till the cast concrete was set. For floors these systems were competitive above the fusée system. Due to the conical end the fusées could be laid easily in curved roofs, thus in practice these elements were used by preference in shells and vaults.For concrete structures the self-weight is often as much as 50% to 75% of the total load. Decreasing the self-weight thus decreases the total load considerately. As a result the depth of the structure can be reduced. In the fifties buildings were not as insulated as is the current practice. Decreasing the thermal conductivity, although minor compared to present standards, was welcomed as it improved comfort and reduced the heating cost in winter. Thanks to the ceramic elements the formwork could be stripped earlier and the time required for the successive cycles of the construction was reduced as well. Nevertheless the vaults had a serious disadvantage, the construction on the building site of concrete roofs is labour-intensive. During the sixties the cost of labour was rising fast and this system could not compete with steel and timber roofs using prefabricated components. Consequently this system became obsolete in the last quarter of the twentieth century.

Technique of construction

The fusées are cylindrical elements with a length of 35 cm, an outward diameter of 8 cm and a wall thickness of 1 cm [Eck54]. To join the elements one of the ends is shaped conically. The conical end is placed into the open rear of a second element. The joints can be rotated slightly to follow the curvature of the roof. The fusées are embedded in mortar or concrete [Lan49]. Probably in practise to save cement concrete composed of cement, sand and gravel was preferred above mortar, composed of cement and sand. Nevertheless to fill the voids between the fusée elements the diameter of the gravel had to be quite small. The spacing between the fusée elements was 1 cm at minimum, so possibly a gravel C2-C8 was used. The construction order was as follows: first scaffolds were erected and the mould was greased, a layer of about 2.5 cm of liquid concrete was poured. Next the elements were wetted and pushed in the non hardened concrete with a twist, starting at the gutter. At the top the elements were connected with a special element with two open ends, known in the Netherlands as a 'mof'. Sometimes a second and a third layer was added. The liquid concrete in the second layer was poured when six rows of fusées were laid on the formwork. The top layer was smoothened with a straightedge. The concrete needed to set for at least 36 hours before the formwork could be stripped [Eck54]. Some contractors used a sliding mould with a width of about 2.0 m, which could be moved on rails, to reuse the formwork many times.

Dimensions

The number of layers was chosen with respect to the span and loads. For barrel vaults one layer of fusées was applied for spans of 15 m at most [Eck54], for larger spans it was recommended to use two or three layers. Actually in practice, as for example for roofs in Woerden and Dongen, barrel vaults were constructed with only one layer of fusées and a span up to 20 m. The thickness of the shell was chosen according to the number of layers. For a shell with one layer a minimum thickness of 11 cm, for a vault with two layers a minimum thickness of 18 cm and for a roof with three layers a minimum thickness of 25 cm was recommended [Eck54].

Reinforcement

Generally the barrel roofs were reinforced, but some vaults were constructed like masonry vaults without reinforcement. According to professor Ros barrel vaults with 3 layers of fusées and a span of 15 m could be constructed without reinforcement (Lan49]. Otherwise Van Eck and Bish recommended to reinforce vaults with a span above 10 m. The reinforcement was selected to match the span and the loads. Generally the centre-to-centre distance of the bars was 18 cm. For a span of 10 m to 15 m a reinforcement of Ø6-18 cm was recommended. For a span above 15 m the reinforcement was increased to Ø8-18 cm [Eck54]. Of course just as in the present the designer had to prove the load bearing capacity of the roof with a calculation.
Originally the forces, stresses and dimensions were expressed in kg and cm. To accommodate the reader this notation is transferred into the modern SI-system using Newton and mm.

Depth of unreinforced vaults

The covering on the fusées had to be at least 10 mm. Thus for a one-layered vault the depth of the vault was at least:

$$t = 80 + 2 \times 10 = 100 \text{ mm.}$$

This depth is less than the minimal depth, 110 mm, described by Van Eck and Bish [Eck54].
For the fusées the minimal spacing for bond to the concrete had to be 10 mm. Thus for roofs composed of two layers of fusées the centre distance of the fusées is minimal 90 mm. If the cover on the fusées was 10 mm then the thickness of the roof had to be minimal:

$$t = 10 + 80 + 10 + 80 + 10 \text{ mm} = 190 \text{ mm.}$$

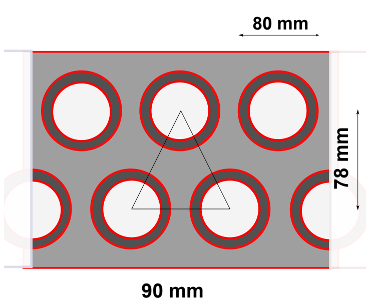


FIGURE 1.3 Section of a vault with two staggered layers of fusée elements.

This depth is larger than the minimal depth described by Van Eck and Bish [Eck54]. Building the elements stepwise can reduce the depth of the vault. The distances of the lines through the centres of the layers is equal to $\frac{1}{2} \times 90 \times \sqrt{3} = 78$ mm, see figure 1.3. The coverage on the fusées is 10 mm, thus the depth of a vault composed with two layers of fusées built stepwise and not reinforced with distribution bars is at least:

$$t = 10 + 40 + 78 + 40 + 10 \text{ mm} = 178 \text{ mm} \approx 180 \text{ mm.}$$

This depth is equal to the minimal depth described by Van Eck and Bish [Eck54].

Depth of reinforced vaults

Possibly the depth had to be increased in case the section was reinforced. According to the building code of 1950 [C1] the minimal cover on the reinforcement had to be at least 10 mm for slabs with a maximum thickness of 120 mm. Generally Fusee Céramique vaults were reinforced with main bars running parallel to the span and fusées. The section with one layer of fusées, showed in figure 1.4 , is reinforced with rebars Ø8-180. The showed section is not reinforced with distribution bars running perpendicular to the span. The thickness of the roof, if the reinforcement running parallel to the span was laid between the fusées, had to be minimal equal to:

t = 10 + 80 + 10 = 100 mm.

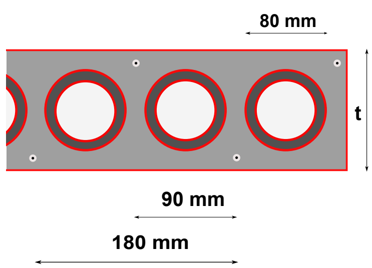


FIGURE 1.4 Section of a vault with one layer of fusée elements reinforced with main rebars positioned parallel to the span.

For roofs composed of two layers of fusées the depth must be increased. The minimal spacing between the fusées was 10 mm, so the centre distance of the fusées is at least 90 mm. According to the building code of 1950 [C1] the minimum cover on the bars had to be 15 mm for slabs with a thickness of 120 mm or more. The thickness of the roof, if the reinforcement running parallel to the span was laid between the fusées, had to be minimal equal to:

t = 15 + 80 + 10 + 80 + 15 mm = 200 mm.

Building the elements stepwise can reduce the depth of the vault. The distances of the lines through the centres of layers is equal to $90\sqrt{3}/2 = 78$ mm, see figure 1.3. The coverage on the rebars is 15 mm, thus the depth of a vault composed with two layers of fusées built stepwise and not reinforced with distribution bars is at least:

t = 15 + 40 + 78 + 40 + 15 mm = 188 mm ≈ 190 mm

Depth of vaults reinforced with rebars parallel and perpendicular to the span

Generally one way-floors and roofs are reinforced with main bars parallel to the span and distribution bars perpendicular to the span. For floors the distribution bars are generally placed in the second layer, so the main bars are positioned on the outside. Generally the diameter of rebars and distribution bars is respectively Ø8 and Ø6 at minimum. Nevertheless a drawing of a fusée vault shows reinforcement with rebars Ø6 and distribution bars Ø4 [Lan49]. Using rebars Ø6 and distribution bars Ø4 into the second layer, the thickness of a vault with one layer of fusées must be al least:

t = 10 + 6 + 4 + 80 + 4 + 6 + 10 = 120 mm.

Using rebars Ø8 and distribution bars Ø6 will increase the depth, then the covering has to be increased to 15 mm and the depth thickness has to be about 140 mm. However if the distribution bars are placed into the first layer and the main bars positioned between the fusées the minimal depth is at minimum:

$$t = 10 + 4 + 80 + 4 + 10 = 108 \approx 110 \text{ mm.}$$

For roofs, composed of two layers of fusées reinforced with main bars between the fusées and distribution bars Ø4 in the first layer, the depth has to be minimal:

$$t = 15 + 4 + 80 + 10 + 80 + 4 + 15 \text{ mm} = 208 \text{ mm} \approx 210 \text{ mm.}$$

Again building the elements stepwise can reduce the depth of the vault. The distances of the lines through the centres of layers is equal to 78 mm, see figure 1.3. The coverage on the distribution bars laid in the first layer is 15 mm; the depth of a vault composed with two layers of fusées built stepwise has to be at least:

$$t = 15 + 4 + 40 + 78 + 40 + 4 + 15 \text{ mm} = 196 \text{ mm} \approx 200 \text{ mm.}$$

§ 1.2 Form and Curvature

Generally the curvature of the barrel vaults was following a segment of a circle, a parabola or a funicular curve. To show the differences the following table and graph show the coordinates for a vault with a ratio rise to span of $f/a = \frac{1}{4}$, with $a = \frac{1}{2}\, l$, for a parabola, a circle segment, a cosine and a catenary.

parabola: $$y/f = x^2/a^2 \qquad [1.1]$$

circle segment: $$y/f = R\,[1 - (1 - x^2/R^2)^{1/2}]/f \quad \text{with } R = \tfrac{1}{2}(a^2 + f^2)/f \qquad [1.2]$$

cosine: $$y/f = 1 - \cos(\tfrac{1}{2}\pi\, x/a) \qquad [1.3]$$

catenary: $$y/f = [\cosh(x/c) - 1]\, c/f \quad \text{with } c = H/q \qquad [1.4]$$

Expression [1.4] describes the catenary for the parameters $c = H/q$ and the rise f. Te Boveldt [Bov94] gives an expression to approach the coefficient c for a ratio f/a:

$$2\,a/c = \frac{4\,f/a}{(1 + \tfrac{1}{2} f^2/a^2)^{1/2}} \qquad [1.5]$$

In practice most Fusée Céramique vaults were designed with a ratio rise to span of $f/a = \frac{1}{4}$ then c is equal to 2.031 a. Probably in the past most engineers, using a sliding rule, designed the catenary with $c = 2\,a$. For $c = 2\,a$ and $f/a = \frac{1}{4}$ the coordinates follows from [1.4]:

$$y/f = 8\,[\cosh(\tfrac{1}{2}\, x/a) - 1]/B \qquad [1.4']$$

Substitute $y = f$ and $x = a$ to define the parameter B: $B = 8\,[\cosh(\tfrac{1}{2}) - 1] = 1.021$

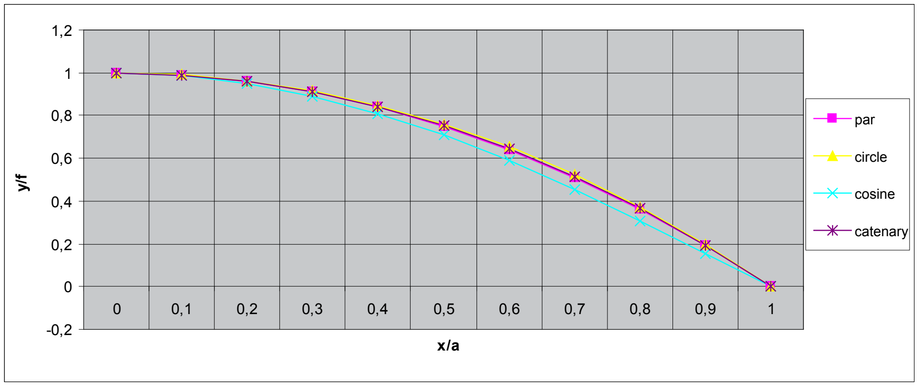


FIGURE 1.5 Curvature following a parabola, circle segment, cosine and catenary

Table 1.1 gives the coordinates according to [1.1], [1.2], [1.3] and [1.4']. Comparing the curvatures, see figure 1.5, shows that the differences between the catenary and parabola are minor. For x/a = 0.7 the difference between the catenary and parabola is Δ/f = 0.0052. For f=*l*/8 the ratio difference to the span is equal to Δ/*l* = 0.00065. Generally the fusée roofs were dimensioned slender. For building Q in Woerden the ratio thickness to the span is 0.0066. for this vault the difference between the catenary and parabola is only $^1/_{10}$ of the thickness.

x/a	Parabola y/f	Circle y/f	Cosine y/f	Catenary y/f	Differences catenary and parabola Δ/f
0	0	0	0	0	0
0.1	0.01	0.00940	0.01231	0.00980	0.0002
0.2	0.04	0.03773	0.04894	0.03921	0.0008
0.3	0.09	0.08513	0.10899	0.08830	0.0017
0.4	0,16	0.15195	0.19098	0.15723	0.0028
0.5	0.25	0.23864	0.29289	0.24614	0.0039
0.6	0.36	0.34586	0.41222	0.35525	0.0048
0.7	0.49	0.47442	0.5460	0.48484	0.0052
0.8	0.64	0.62536	0.69098	0.63524	0.0048
0.9	0.81	0.80000	0.84357	0.80682	0.0032
1.0	1.0	1.0	1.0	1.00001	0

TABLE 1.1 Coordinates y/f for a parabola, circle segment, cosine and catenary

The catenary can be approached with the following expression:

$$y = (x^2/c)/2 + (x^4/c^3)/24 + \ldots\ldots (x^m/c^{m-1})/(m!)$$

For a vault with f/a = ¼ the parameter c is approached with to c = 2 a.

$$y/f = [(x^2/a)/4 + (x^4/a^3)/192 + \ldots\ldots]\, a/f$$

Substituting f/a = ¼ gives: $y/f = (x/a)^2 + (x/a)^4/48 + \ldots\ldots$ [1.6]

Notice that if only one term is used and the following terms are neglected expression [1.6] is exactly equal to expression [1.1] describing a parabola. The second term is very small, in practice the third and following terms can be neglected.
Due to an equally distributed load a parabolic vault is subjected to normal forces only and not loaded by bending, so a parabola is quite efficient for this loading. Actually the dead load acting on an arch

is equally distributed over the surface, due to this loading the structure is subjected to normal forces only in case the curvature follows a catenary. Thus structurally the catenary transfers the dead load very efficiently. Unfortunately the calculation of a vault following a catenary is significantly more complicated, especially in the pre-personal computer era. For low rise vaults the differences between the parabola and catenary are very small so in the past most engineers preferred the parabolic form, which can be calculated much more easily.

FIGURE 1.6 Roof lights illuminated the inner space of building Q in Woerden and partitioned the vault. Photo taken by author

The barrel vaults could be supported by the foundations, walls or frames, composed of beams and columns. For vaults resting directly on the foundations the horizontal thrust could be resisted by the foundations or a reinforced concrete ground floor. For vaults resting on frames or walls the thrust was resisted with steel bars, generally with a centre-to-centre distance of about 1.0 m. The connection of the bars with the roof had to be detailed very carefully. Preferentially the tension rods were connected at the centre of the beams to avoid twisting moments [Toe53]. To illuminate the inner spaces, roof lights could be added, running parallel to the span of the roof, as shown in figure 1.6.

§ 1.3 Vernacular vaults composed of Guna-Tubes

In Auroville, India, small vaults, the Wardha roofs, are built with burnt clay pipes, the so called Guna-Tubes. The technique of construction is labour intensive. The following description summarizes the text of Scienceandsociety [13]. The tapered conical clay pipes are made by local craftsmen and burnt in a small oven. Firstly a skeleton of steel pipes is erected on top of the bearing walls. The Guna-Tubes are laid on the formwork; the conical top is pushed in the open end of the neighbouring element. The arches are laid in reverse direction to minimize the gaps. To joint the elements and to create watertight roof cement plaster is constructed on the top of the vault. The inner side is not finished. The mould of the vault is composed of the Guna Tubes supported by the steel pipes. The mortar is resting on the Guna-Tubes only. The clay pipes, supported by the frame of steel bars, are stiff and strong enough to support the top layer till the concrete is set. The frame of steel pipes is removed when the top layer is set, often within 12 hours. Probably the roof sinks slightly when the frame of steel pipes is removed to enable the elements to joint firmly into the sockets. The vaults are labour intensive, but the cost of materials and mould are small. The vaults are not reinforced, nevertheless these structures are said to be safe even in earthquake prone areas.

§ 1.4 The bovedas of Eladio Dieste

The buildings, which were constructed with the Fusée Céramique system, resemble the buildings, the so called bovedas, made by Eladio Diëste (1917-2000), who built many remarkable shells, constructed with bricks, mostly in Uruguay and Spain.Diëste preferred brick above concrete for the following reasons [Die87]:

- Elevated Mechanical resistance; between 50 and 100 MPa and up to 150 MPa;
- Lightness unachievable with reinforced concrete;
- Less elasticity than reinforced concrete, which gives the structure greater adaptability to deformations.
- Good aging with minimal upkeep;
- Repairs and changes less noticeable;
- Good thermal insulation, incremented even more by introducing holes;
- Better acoustic behaviour;
- "Natural" regulation of environmental humidity;
- Less heat radiation;
- Incomparable lower price per cubic meter;
- Saving costs.

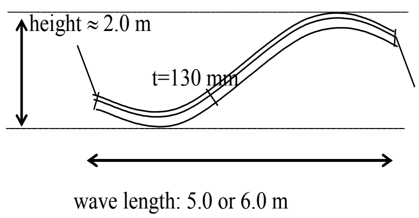


FIGURE 1.7 Section of a discontinuous 'Gausa' vault. The generator curve follows a sine. The directive perpendicular to the section is a catenary.

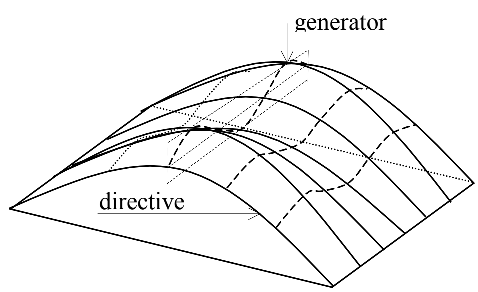


FIGURE 1.8 Section of a continue 'Gausa' vault. The generator curve follows a sine. The directive parallel o the span is a catenary.

Diëste constructed barrel vaults (the so called horizontal silos), barrel shells and the 'Gausa' vaults, which were undulated longitudinally, to increase the stiffness and prevent buckling. These double curved vaults are described using a generator, which is moved along a line, the directive. For a barrel vault the generator is a straight line, which is moved along a curved line, mostly a catenary, a parabola or a part of a circle. The generator of a 'Gausa' vault is a curve, mostly a sine, with a wavelength of 5.0 to 6.0 m and a rise of about 2.0 m, see figure 1.7. The directive is curved and follows mostly a catenary, parabola or a segment of a cycle, see figure 1.8.

The span of the directive of the 'Gausa' vaults varies from 20 m to 50 m, the width varies from 4.0 to 6.0 m and the thickness varies from 120 mm to 150 mm. The shells were composed of massive or hollow core bricks, 250 × 250 × 73 mm, and a top layer of 30 – 70 mm reinforced with a mesh Ø4.2 –150 mm. The voids were reinforced with bars Ø6-190 mm in both directions [Ver07]. Often the top layer was reinforced with steel oblong rings, which were post-tensioned by pulling two opposite midpoints to each other.
For the 'Gausa' vaults Diëste preferred the catenary so only the permanent surface load compresses the structure. The design of a vault following a catenary is complex;. Due to the varying height of the generator curve, the rise of the funicular curves of the directives varies. Further the rise of a funicular curve depends on the load and thrust, so the description of the roof is difficult. To describe the funicular curvatures of the

spans, tables were made with a computer at the faculty of engineering of the Universidad de Montevideo in 1967 [Die87].

Construction

The construction was as follows, first the formwork was erected, the bricks were placed on the mould, and the reinforcement was placed in the voids as well as in the top layer above the bricks. The voids were filled with a mortar of cement and sand. Next the top layer, resting on the bricks, was poured. The shells were finished with white paint, to waterproof the roof and reflect the sun. The moulds, needed for the 'Gausa' vaults are an expensive investment, but repeated use decreases the lifecycle costs. The structure of the moulds is iron covered with wood. Mechanical jacks are used to raise and lower them smoothly [Die87]. The formwork was replaced rather quickly. Diëste wrote: "about three hours of hardening time for mortar is necessary in 15 m vaults and 14 hours for 50 m vaults" [Die87]. This short cycle time could be realized because the mortar in the voids was only 20 mm thick, the quality of the mortar was good and the water, added to the mortar, was partly absorbed by the bricks [Ver07]. For petrol stations, market halls and bus stops Diëste designed barrel shells with huge cantilevers. Due to the cantilevers the vaults are subjected to bending causing tensile stresses in the top layer. To resist the tensile stresses the vaults were post-tensioned with cables laid in the top layer of the vault. The cables were tensioned by contracting the loop, which caused compressive stresses to compensate the tensile bending stresses. For the omnibus-station "de Salto" Diëste designed barrel shells with a cantilever of 13 m, supported by rows of columns with a centre-to-centre distance of of 5.75 m. For omnibus-station Turlit the cantilever was increased to 14 m, the five barrel shells are supported by rows of columns with a centre-to-centre distance of 5.75 m.

§ 1.5 Comparing the bovedas and the Fusée Céramique vaults

Comparing fusée roofs and bovedas we can notice that with both techniques costs were reduced by using ceramic elements to save on cement and reinforcement. Also the technique of construction is similar; elements are joined by the hardening of the mixture. Generally both systems were reinforced with steel rebars.
However the elements of both systems are fundamentally different: the fusées are cylindrical elements, embedded in concrete and jointed by placing the conical top into the rear of the next element; for the bovedas the bricks were jointed by the mortar poured in the voids between the bricks. The Fusée Céramique cylindrical vaults are form-active structural systems composed of reinforced concrete and ceramic elements. The bovedas are surface-active structural systems. The bricks are connected through the reinforced voids and the top layer, which could be post tensioned too, for example to realize a cantilever. The voids filled with mortar form a grid, which can be described as a crossbeam system of reinforced concrete bars.
Both systems offer the designer a large free span. The span of the single curved Fusée Céramique vaults was up to 24 m, the centre-to-centre distance of the columns supporting the edge beams was about 5 to 6 m and the thickness varied from 110 to 290 mm. For example the roof of the N.V. Twentsche Stoombleekerij in Goor was composed of three fusée vaults with a span of 16.3 m each and a length of about 25 m [Toe53].
The span of the double curved bovedas varies from 20 to 50 m; the wavelength of the undulation needed to stiffen the roof and prevent buckling varies from 4.0 to 6.0 m, the thickness varied from 120 to 150 mm. For example Diëste designed for the Agro-industrias Massaro barrel shells with a

span of 16.4 m, supported by columns with centre-to-centre distances varying from 30 to 35 m and a cantilever of 14 m [Die87].
Structurally due to the undulation the forms of the double curved surfaces of the bovedas are stiffer and stronger than the single curved vaults constructed with fusées, consequently the span of the bovedas could be larger and the thickness lower. Actually the use of the fusées was not restricted to barrel vaults and could be used for double curved surfaces too, several domes were built with fusées, but generally the designers preferred a single curved vault to simplify the formwork. Thanks to the undulated form the roofs made by Diëste are much stiffer than the single curved vaults. The barrel vaults made in the Netherlands with the Fusée Céramique system are very slender, for some of these roofs the resistance against buckling was sometimes poor, this will be demonstrated in the following chapters.
Due to asymmetric loads form-active structure systems are subjected to bending moments. Barrel vaults of masonry of concrete will crack when the bending stresses are larger than the normal compressive stresses. The fusée vaults could be reinforced on both sides above and below, which was effective for the asymmetrical live loads. Diëste designed the bovedas mostly with a catenary, so these vaults were compressed by the dead load; furthermore he stiffened the vaults with the longitudinal undulation to prevent buckling. To resist bending due to variable loads steel bars were positioned in the voids. For the cantilevers cables were laid in the top layer to post-tension the vaults.

§ 1.6 Conclusions

Generally the curvature of a low rise Fusée Céramique vault was following a parabola or catenary, the ratio of the rise to span was $^1/_8$ and the thickness was varying from 100 mm to130 mm for vaults with one layer of fusées and varying from 180 mm to 210 mm for vaults with two layers of fusées. Comparing a parabola, cosine, circle segment and catenary shows that for these low rise vaults the differences between these curves are very small. The following chapters will analyse the effect of the curvature for the load transfer.
The fusée roofs were made in the same era as the shells designed by Diëste. The systems were developed simultaneously in France and Uruguay. Nevertheless in spite of the important differences both systems outwardly resemble each other. Both systems were used regionally. This induces the question of why these systems were not applied worldwide. The buildings constructed by Diëste were described in 1987 [Die87]. Possibly the technique was applied outside Uruguay too if the technique was described earlier. In the Netherlands just after WWII ceramic systems were not very rare, many systems were available and used, mostly for floors in residential apartment houses. For floors the Fusée Céramique system was not competitive to systems using beams and infill that did not need expensive formwork, but for curved roofs this system was competitive in the fifties and sixties of the XXth century for barrel vaults and domes with a span from 10 m up to 24 m. However in India the Guna Tube roofs are still made, apparently hybrid systems of ceramic tubes and concrete finishing are still competitive if the cost of labour is low.

The following chapters describe the structural aspects. Chapter 2 reconstructs the design of a vault as described in the past. Chapter 3 analyses the second order and critical buckling load for arches and vaults. Chapter 4 studies the time dependant effects of vaults composed of fusées, reinforcement and steel. The structural design of a vault is analysed in chapter 5. Chapter 6 describes methods to strengthen parabolic vaults. Chapter 7 shows for these vaults the effect of the strengthening. Chapter 8 discusses the sustainability and construction of infill elements. Chapter 9 analyses the structural design of prefabricated circular vaults strengthened with slender steel elements. Chapter 10 describes the tests to validate the structural concepts. The conclusions and recommendations are given in chapter 11.

2 Analysis of the design of a Fusée Céramique vault in the past

In this chapter the methods of structural design for fusée structures are discussed. To introduce the fusée system articles were written in technical journals and books. This chapter follows the design of a roof as described by Van Eck and Bish in the journal Cement [Eck54]. The description of the calculations was quite condensed. Some information, probably assumed to be clearly evident to the readers half a century ago, was taken as granted and not explained but for modern readers many of the assumptions used are not as clearly evident. To learn from the past it is necessary to understand how the area, stiffness and stresses were calculated. To understand the methodology the details missing in the paper have been researched and added to the description. Originally the forces, stresses and dimensions were expressed in kg and cm. To accommodate the reader this notation is transferred into the modern SI-system.

§ 2.1 Structural design and validation of a vault in 1955

The design of a parabolic vault, described by Van Eck and Bish in Cement [Eck54], is used to show the theories and concepts current fifty years ago. The calculation is reconstructed step-by-step conforming to the theories practiced during the nineteen fifties. According to the given calculation the reconstruction has the following stages:

- Defining the geometry and materials;
- Analysis of the forces and moments;
- Analysis of the second order effects and buckling force
- Analysis of the stresses.

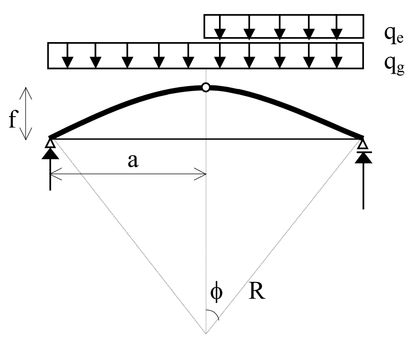


FIGURE 2.1 Scheme of the vault subjected to an equally distributed permanent and asymmetrical live load.

Loads

The vault has a span of l = 24 m, half of the span a = ½ × 24 = 12 m and a rise of f = 3.0 m. The thickness of the vault is equal to 200 mm. The structure has to resist a live load p_e =1.0 kN/m^2 and a permanent load p_g =3.30 kN/m^2.

The permanent load is caused by the dead weight, finishing and ceiling:

Dead weight:	2.90 kN/m^2
Finishing:	0.15 kN/m^2
Ceiling:	0.25 kN/m^2
Permanent load:	3.30 kN/m^2

Thrust

According to the Theories of Mechanics the thrust is calculated with the following expression:

$$H = {}^{1}/_{8}\, q\, l^2/f \qquad [2.1]$$

For a permanent and live load q = 4.3 kN/m, f = 3.0 m and l = 24 m the thrust is equal to:

$$H = {}^{1}/_{8} \times 4.3 \times 24^2/3.0 = 103.2 \text{ kN.}$$

Section

In a section with a width of 1.0 m eleven fusées were placed with a spacing of 10 mm. Firstly a layer of liquid concrete with a thickness of about 25 mm was poured, next the fusées were pushed about 10 mm into the concrete. The coverage on the fusées was approximately 15 mm. The centre-to-centre distance between the staggered elements is with a spacing of 10 mm equal to 80 + 10 = 90 mm. If the fusées were built stepwise then the centre-to-centre distance of the two layers had to be at least ½ × 90 × √3= 78 mm, see figure 1.2. Thus the thickness of the vault with fusées built step wise had to be at least:

$$t > 2 \times (15 + ½ \times 80) + 78 = 188 \text{ mm} \approx 190 \text{ mm.}$$

Otherwise with two layers of fusées, positioned above each other, the thickness of the vault has to be at least:

$$t \geq 15 + 80 + 10 + 80 + 15 = 200 \text{ mm.}$$

For the described vault a thickness of 200 mm was chosen. The fusées can be built in a staggered manner as well as with the centres positioned in a vertical line.
The vault was reinforced with bars Ø8 – 180 mm in the top and bottom. In the past the Dutch code, GBV 1950 [C2], requires for floors with a thickness of 120 mm or more a cover on the reinforcement of 15 mm minimal. Van Eck and Bish did not mention any distribution bars; probably this vault was not reinforced with distribution bars. In practice the Fusée Céramique vaults were often partitioned by dilatations with a centre-to-centre distance of about 5.0 m. Vaults, spanning in one way, are mainly subjected to a normal load and a relatively small bending moments, acting parallel to the span, consequently the bending stresses acting perpendicular to the span are very small. The Fusée Céramique vaults can be compared with prefabricated pre-tensioned hollow core elements, which are frequently used in the Netherlands. These elements are also not reinforced with distribution bars.
According to the GBV 1950 the cover on the main bars Ø8 had to be 15 mm, probably the main bars are positioned between the fusées.
For the concrete, steel and fusées the area and second moment of the area is calculated and shown in the following tables for a part of the roof with a width of 1.0 m. The vault is assumed to be not reinforced with distribution bars. The main bars are laid between the fusées with a covering of 15 mm. The assumption is made that for the calculation of A_c and I_c the reduction due to the area of the rebars was neglected.

			Area [mm^2]
Fusées:	A_f =	$22 \times ¼ \pi \times (80^2-60^2)$ =	48.4×10^3 mm^2
Concrete:	A_c =	$200 \times 1000 - 22 \times ¼ \pi \times 80^2$ =	89.4×10^3 mm^2
Rebars upper side:	A_s =	$¼ \pi \times 8^2 \times 1000/180$ =	279 mm^2
Rebars, lower side:	A_s =	$¼ \pi \times 8^2 \times 1000/180$ =	279 mm^2

TABLE 2.1 Area of the fusées, concrete and steel for a section of the vault with b = 1.0 m.

		Second moment of the area	
Concrete:	I_c =	$1000\times200^3/12 - 22\times¼ \pi\, 80^2\times45^2 - 22\times\pi \times80^4/64$ =	399×10^6 mm^4
Fusées:	I_f =	$22 \times \pi \times(80^4- 60^4)/64+ 22\times ¼ \pi \times (80^2- 60^2)\times45^2$ =	128×10^6 mm^4
Steel:	I_s =	$2 \times 279 \times (½ \times 200 - 15 - ½ \times 8)^2$ =	3.7×10^6 mm^4

TABLE 2.2 Second moment of the area of the fusées, concrete and steel for a section of the vault with b = 1.0 m.

Stresses and stiffness

The authors asserted that due to the load the vault is subjected to a normal stress of σ_c = 0.77 MPa, but did not describe the calculation of the stresses. In the past the assumption was made that the fusées and concrete were well connected. The demolition of building Q in Woerden showed that nearly all fusées were well bond to the concrete, only one element survived. According to the Theory of Elasticity the stresses acting in the sections of the structure composed of concrete, fusées and reinforcement follows from:

$$\sigma_i = \frac{N E_i}{EA} \qquad \text{with: } EA = E_c A_c + E_f A_f + E_s A_s \qquad [2.2]$$

To calculate the stress we have to to define Young's modulus of the section of the steel, fusées and concrete. The Young's modulus of steel is $E_s = 2.1 \times 10^5$ MPa. The stiffness of a concrete structure depends on many variables such as compressive strength, shrinkage, creep and cracks. In the fifties generally the stresses in concrete and reinforcement were calculated with a ratio of the Young's modulus for steel and concrete of $n_s = E_s/E_c$. In the article the stiffness of the fusées was not specified. According to experiments of Ros [Lan49] the stiffness of a fusée element is equal to: E_f = 27500 MPa. Nevertheless the deformations of the joints will affect the stiffness of the string, so for the calculations the stiffness of a string of fusées is smaller than the stiffness of a single element. Probably the designers used for calculations a reduced value: E_f < 27500 MPa. For fusee roofs in Dongen Bish used a stiffness of the fusées and concrete of respectively equal to E_f = 17000 MPa and E_c = 21000 MPa, thus n_s = 10 and n_f = 0.81. According to the Theory of Elasticity the stiffness can be defined with:

$$EI = E_c I_c + E_f I_f + E_s I_s \qquad \rightarrow \qquad EI = E_c (I_c + n_f I_f + n_s I_s) \qquad [2.3]$$

Substituting the values for E_c, I_c, I_f and I_s into [2.3] gives for the vault that is not reinforced with distribution bars:

$$EI = 2.1 \times 10^4 \times (399 \times 10^6 + 0.81 \times 128 \times 10^6 + 10 \times 3.7 \times 10^6) = 11.3 \times 10^{12} \text{ Nmm}^2$$

The stiffness given in the paper [Eck54] is equal to EI = 12×10^{12} Nmm^2, the difference is about 6%. In the past calculations were made with a sliderule, and thus were less accurate than present calculations made with calculators or computers. Due to the truncations the differences between the former calculations and the calculations of the present are unavoidable. Further it is possible that the calculation of the second moment of the area of concrete I_c and fusées I_f, was simplified by neglecting the effect of respectively $n \pi D^4/64$ and $n \pi (D^4- d^4)/64$, see table 2.3.

		Second moment of the area	
Concrete:	I_c =	$1000 \times 200^3/12 - 22 \times ¼ \pi\, 80^2 \times 45^2$ =	443×10^6 mm^4
Fusées:	I_f =	$22 \times ¼ \pi\, (80^2 - 60^2) \times 45^2$ =	98×10^6 mm^4
Steel:	I_s =	$2 \times 279 \times (½ \times 200 - 15 - ½ \times 8)^2$ =	3.7×10^6 mm^4

TABLE 2.3 Simplified calculation of the second moment of the area of the fusées, concrete and steel, neglecting $n\,\pi\, D^4/64$ for I_c and $n\,\pi\,(D^4 - d^4)/64$ for I_f

Then the stiffness is equal to:

$$EI = 2.1 \times 10^4 \times (443 \times 10^6 + 0.81 \times 98 \times 10^6 + 10 \times 3.7 \times 10^6) = 11.7 \times 10^{12}\ \text{Nmm}^2$$

$$EI \approx 12 \times 10^{12}\ \text{Nmm}^2$$

Calculation of the normal stresses.

According to the Theory of Elasticity the stress in a section, composed of fusées, concrete and steel, subjected to a normal force N is equal to:

$$\sigma_x = \frac{N\,E_x}{E_c A_c + E_f A_f + E_s A_s} \qquad [2.4]$$

With σ_x is the stress in either the concrete, fusées or steel. Substituting $n_f = E_f/E_c$ and $n_s = E_s/E_c$ into this expression gives the following formulae to calculate the stresses in the concrete, fusées and reinforcement:

$$\sigma_c = \frac{N}{A_c + n_f A_f + n_s A_s} \ ; \quad \sigma_f = \frac{n_f N}{A_c + n_f A_f + n_s A_s} \ ; \quad \sigma_s = \frac{n_s N}{A_c + n_f A_f + n_s A_s}$$

With:
$E_c = 2.1 \times 10^4$ MPa; $E_f = 1.7 \times 10^5$ MPa; $n_f = E_f/E_c = 0.81$; $E_s = 2.1 \times 10^5$ MPa; $n_s = E_s/E_c = 10$; $A_c = 89.4 \times 10^3$ mm^2; $A_f = 48.4 \times 10^3$ mm^2; $A_s = 2 \times 279$ mm^2.
Substituting these features into the formulae results for a force N in:

$$\sigma_c = \frac{N}{89.4 \times 10^3 + 0.81 \times 48.4 \times 10^3 + 10 \times 2 \times 279} = \frac{N}{134184}$$

$$\sigma_f = \frac{0.81\,N}{134184} \quad \text{and} \quad \sigma_s = \frac{10\,N}{134184}$$

For the full load the thrust force is equal to H = 103.2 kN. The stress in the concrete is at the top of the vault, with N =H, equal to:

$$\sigma_c = \frac{103200}{134184} = 0.77\ \text{MPa}$$

This stress is equal to the stress given in the article. Probably this reconstruction describes the method used by the authors and the assumptions sufficiently accurate. Evidently the stress will increase to the supports .

Buckling

The critical buckling load was calculated next using the following equation:

$$q_{cr} = \frac{EI\,[\pi^2/\phi^2 - 1]}{R^3} \qquad [2.5]$$

This formula is described by Timoshenko [Tim52] and was defined for a circular arch for a radial load. For a shallow arch the curvature of a parabola approaches the curvature of a circle segment quite well, so to calculate the critical buckling force the authors approach the parabola as a segment of a circle with a radius R. For an arch, subjected to a radial load, the normal force is equal to N = q R. Thus the critical buckling force fis or an arch, subjected to an equal distributed radial load: $N_{cr} = q_{cr}$ R. Substituting q_{cr} [2.5] leads to:

$$N_{cr} = \frac{EI\,[\pi^2/\phi^2 - 1]}{R^2} \qquad [2.6]$$

For a circle segment the radius is constant; for a parabola the radius increases from the top to the supports. For the vault with a span *l* = 2 a, the radius at the supports was calculated with:

$$R = a/\sin\phi \qquad [2.7]$$

For the parabolic vault the angle ϕ is equal to 2 β; β is the angle between the diagonal, running from the crown to the support and the horizontal line through the supports. This angle was calculated with:

$$\tan\beta = f/a \qquad [2.8]$$

With f = 3.0 m, *l* = 24 m and a = 24/2 = 12 m the angle β follows from: $\tan\beta = f/a = 3.0/12 = 0.25$; thus $\beta = 0.245$ radians, $\phi = 2\beta = 0.49$ radians and $\sin\phi = 0.47$.

The radius of the circle segment is equal to: $R = a/\sin\phi = 12/0.47 = 25.6$ m.

The stiffness of the vault was given: $EI = 12 \times 10^{12}$ Nmm². Substituting R, EI and ϕ into the expression [2.6] gives the following buckling load:

$$N_{cr} = \frac{12 \times 10^{12} \times [\pi^2/0.49^2 - 1]}{25600^2} = 734 \times 10^3 \text{ N}$$

Actually the critical buckling load can be calculated more precisely as follows. The radius of a parabola varies, for a parabola the radius is equal to:

$$R_\phi = \frac{a^2\,(1 + 4\,f^2/a^2)^{1/2}}{2\,f} \qquad [2.9]$$

Substituting the rise f and half of the span a = ½ × 24 = 12 m gives:

$$R_\phi = \frac{12^2\,(1 + 4 \times 3^2/12^2)^{0.5}}{2 \times 3} = 26.83 \text{ m}$$

For the parabolic vault the angle ϕ between the tangent and the horizontal line through the supports can be calculated accurately with:

$$\tan\phi = 2\,f/a \qquad [2.10]$$

With f = 3.0 m, l = 24 m and a = 12 m the angle ϕ follows from: $\tan\phi = 2 f/a = 2 \times 3.0/12 = 0.5$
Thus $\phi = 0.464$ radians.
The critical buckling force N_{cr} is equal to:

$$N_{cr} = \frac{EI\,[\pi^2/\phi^2 - 1]}{R^2} = \frac{12 \times 10^{12} \times [\pi^2/0.464^2 - 1]}{(26.83 \times 10^3)^2} = 747.5 \times 10^3 \text{ N}$$

This buckling force is slightly larger than the buckling force calculated for a circle segment. Next the ratio n was calculated by dividing the critical load N_{cr} by the normal force N:

$$n_{cr} = N_{cr}/N \qquad [2.11]$$

The ratio n_{cr} given by the authors was equal to: $n_{cr} = 6.8$. Unfortunately the authors did not describe the magnitude of the buckling force and the calculation of the normal forces, to check the calculation we have to define the normal forces firstly.

Full load, normal force and buckling ratio

The Forces acting on the vault were calculated with the Theory of Elasticity. The vault was schematised as an arch supported with two simple supports. The stiffness of the supports was neglected; actually the ties will lengthen so the supports will move sidewards. The effect of the assumptions made will be analysed and discussed later.

The vault is subjected to the dead load $q_g = 3.3$ kN/m² and a live load $q_e = 1.0$ kN/m². Due to the symmetrical load q = 3.3 + 1.0 = 4.3 kN/m, the vault is subjected to compression.
Vertical reaction force acting at the support: $V = ½\, q\, l = ½ \times 4.3 \times 24 = 51.6$ kN

Horizontal reaction force acting at the support: $H = \frac{q\, l^2}{8 f} = \frac{4.3 \times 24^2}{8 \times 3} = 103.2$ kN

The normal force acting at the supports is equal to the sum of the vectors V and H:

$$N = (H^2 + V^2)^{0.5} = 115.4 \text{ kN}$$

For x = ½ a = 6.0 the normal force follows from:

$$N = [N^2 + (q\, x)^2]^{0.5} = [103.2^2 + (4.3{\times}6)^2]^{0.5} = 106.4 \text{ kN}$$

For the full load the normal force acting just above the support is equal to N = 115.4 kN. The buckling load calculated in the article is equal to $N_{cr} = 734$ kN, so for the full load the buckling ratio is equal to:

$$n_{cr} = N_{cr}/N = 734/115.4 = 6.4$$

This value is smaller than the value given by the authors: $n_{cr} = 6.8$. Possibly the buckling ratio was calculated for the average value of normal force acting halfway the support and top of the vault. The normal force acting halfway the top at x = 6.0 m, is equal to N = 106.4 kN. Then the buckling ratio is equal to:$n = N_{cr}/N = 734/106.4 = 6.9$. Van Eck and Bish asserted that for the vault the buckling load could be determined with the expression of Euler:

$$N_{cr} = \frac{\pi^2 EI}{l_c^2} \qquad [2.12]$$

For the vault the buckling length l_c is was assumed to be equal to the length of the arch between the support and the top. Further the authors wrote that for the buckling ratio then the maximum normal force N had be used acting just above the support. Actually nowadays for arches the buckling length l_c is assumed to be equal to ψ s, with s is the length between the top and support and $\psi > 1.0$. For low rise arches ψ is only slightly larger than 1.0. This will be shown in chapter 3. Van Eck and Bish give for the buckling ratio n_{cr}, calculated with expression [2.11]: $n_{cr} = N_{cr}/N = 6.7$. Probably this value of n was calculated with a length of a circular segment $s = R\,\phi$. Substituting $\phi = 0.49$ radians and $R = 25.6$ m gives:

$$s = R\,\phi = 25.6 \times 0.49 = 12.5 \text{ m}:$$

$$N_{cr} = \frac{\pi^2 EI}{l_c^2} = \frac{\pi^2 \times 12 \times 10^{12}}{(12.5 \times 10^3)^2} = 758 \times 10^3 \text{ N}; \quad n_{cr} = N_{cr}/N = 758/115.4 = 6.6$$

This value is slightly smaller than the value given by the authors. Probably this difference is only numerical and caused by truncation inherent by the use of a sliding ruler. Actually, as showed in paragraph 6, an accurate expression to define the length of the parabolic vault is:

$$s = f\,(1 + \tfrac{1}{4} a^2/f^2)^{1/2} + \tfrac{1}{4}\,(a^2/f) \times \ln\{2\,f/a + (4\,f^2/a^2 + 1)^{1/2}\} \qquad [2.13]$$

With $a = \tfrac{1}{2}\,l = 12$ m and $f = 3.0$ m:

$$s = 3.0 \times (1 + \tfrac{1}{4} \times 12^2/3.0^2)^{1/2} + \tfrac{1}{4} \times (12.0^2/3.0) \times \ln\{2 \times 0.25 + (4 \times 0.25^2 + 1)^{1/2}\} = 12.48 \text{ m}$$

The difference is very small, for a shallow vault the length of the parabola can be calculated as for a circular segment.

Asymmetric load, normal forces, second order and stresses

An asymmetric load, such as wind , subjects the vault to bending. The vault is assumed to be subjected to a live load of $q_e = 1.0$ kN/m² acting asymmetrically at one side. The permanent load is equal to $q_g = 3.3$ kN/m². The vault is subjected to a minimum load $q_g = 3.3$ kN/m² at one side and a maximum load equal to $q_g + q_e = 4.3$ kN/m² at the other side. The expressions for an asymmetrical load are given in chapter 6.

The vertical and horizontal reaction force acting on the supports are respectively:

$$V_A = \tfrac{1}{2}\,q_g \times l + \tfrac{1}{4}\,q_e \times \tfrac{1}{2}\,l = \tfrac{1}{2} \times 3.3 \times 24 + \tfrac{1}{4} \times 1.0 \times \tfrac{1}{2} \times 24 = 42.6 \text{ kN}$$

$$V_B = \tfrac{1}{2}\,q_g \times l + \tfrac{3}{4}\,q_e \times \tfrac{1}{2}\,l = \tfrac{1}{2} \times 3.3 \times 24 + \tfrac{3}{4} \times 1.0 \times \tfrac{1}{2} \times 24 = 48.6 \text{ kN}$$

$$H = \frac{q_g \times l^2}{8 \times f} + \frac{q_e \times l^2}{16 \times f} = \frac{3.3 \times 24^2}{8 \times 3} + \frac{1.0 \times 24^2}{16 \times 3} = 91.2 \text{ kN}$$

The resulting normal forces acting at the supports are respectively:

$$N_A = (H^2 + V^2)^{0.5} = (91.2^2 + 42.6^2)^{0.5} = 100.7 \text{ kN}$$

$$N_B = (H^2 + V^2)^{0.5} = (91.2^2 + 48.6^2)^{0.5} = 103.3 \text{ kN}$$

The bending moment is equal to:

$$M_o = q_e \times l^2/64 = 1.0 \times 24^2/64 = 9.0 \text{ kN/m}$$

For $x = \tfrac{1}{2}\,a = 6.0$ the normal force follows from:

$$N = [H^2 + (V_A - q_x x)^2]^{0.5} = [91.2^2 + (42.6 - 4.3 \times 6)^2]^{0.5} = 92.7 \text{ kN}$$

The bending moments are increased by the second order effect due to the normal force and deformation, Van Eck and Bish give the following expression to include the second order effect:

$$M = M_o + \Delta M = M_o + \frac{N\,\Delta\,n_{cr}}{n_{cr} - 1} \qquad [2.14]$$

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To define the deformation Δ, the curve of the deformation was represented as a sinusoidal curve and the deformation was calculated with:

$$\Delta = \frac{M_o l^2}{4 EI \pi^2} \qquad [2.15]$$

The increase in the bending moment is equal to: $\Delta M = \frac{N \Delta n_{cr}}{n_{cr} - 1} = 1.14$ kNm

The bending moment including second order is equal to: M = 9.0 + 1.14 = 10.14 kNm.

According to the Theory of Elasticity the normal stress in the concrete is calculated with expression [2.4]:

$$\sigma_c = \frac{N E_c}{E_c (A_c + n_f A_f + n_s A_s)} .$$

Substituting $E_c = 2.1 \times 10^4$ MPa; $E_f = 1.7 \times 10^5$ MPa; $n_s = E_f / E_c = 0.81$; $E_s = 2.1 \times 10^5$ MPa; $n_s = E_s / E_c = 10$, $A_c = 89.4 \times 10^3$ mm², $A_f = 48.4 \times 10^3$ mm², $A_s = 2 \times 279$ mm²:

$$\sigma_c = \frac{N}{89.4 \times 10^3 + 0.81 \times 48.4 \times 10^3 + 10 \times 2 \times 279} = \frac{N}{134184} .$$

For the asymmetrical load the normal force acting in the section halfway the top for x = 6.0 m, is equal to N = 92.7 kN: $\sigma_c = 92700 / 134184 = 0.69$ MPa

The bending stress is calculated with: $\sigma_c = M z E_c / EI$ [2.16]

Where M = 10.14 × 10⁶ Nmm; z = ½ × 200 mm; $E_c = 21 \times 10^3$ MPa and EI = 12 × 10¹² Nmm². The bending stress is equal to:

$$\sigma_c = \frac{10.14 \times 10^6 \times 100 \times 21 \times 10^3}{12 \times 10^{12}} = 1.78 \text{ MPa}$$

The resulting stresses due to the normal force and bending are equal to: $\sigma = 0.69 \pm 1.78$ MPa
The maximum compressive stress is equal to: $\sigma = -2.47$ MPa
The maximum tensile stress is equal to: $\sigma = +1.09$ MPa

These stresses match well with the stresses given in the article: $\sigma = -2.47$ and +1.09 MPa.

The stresses are pretty low. To resist the tensile stresses the vault has to be reinforced with reinforcement steel bars. The calculation of the reinforcement was not described.

§ 2.2 Ultimate stresses

The maximum stresses calculated in the article are very low and well below the ultimate stresses given in the codes of that time. The authors did not explain why the maximum in service stresses had to be pretty small, nevertheless it was good practice to design the structure in this way and considerably limit the stresses. Firstly the ultimate stresses given in the past are described. Langejan gives the following allowable stresses [Lan49] as shown in table 2.4.

			Maximal allowable stress according to Langejan	Maximal allowable stress GBV1955
concrete	compressive bending stress	f_{cu} =	5.0 MPa	8.0 MPa
concrete	compressive normal stress	f_{cu} =	5.0 MPa	6.0 MPa
concrete	shear stress	τ_u =	0.4 MPa	0.5 MPa
steel QR24:	compressive and tensile stress	f_{su} =	140 MPa	140 MPa

TABLE 2.4 Allowable stresses for concrete and steel accordng to Langejan [Lan49] and the GBV1950 [C1]

Generally engineers have to design structures according to the building codes. Table 2.4 shows the ultimate stresses according to the GBV 1950 [C1]. For the fusées the strength was not mentioned in the GBV 1950 [C1]. Langejan asserted that experiments showed for the fusées an ultimate compressive stress equal to f_{fu} = 100 MPa and a maximum bending stress equal to f_{fu} = 13 MPa [Lan49]. For the design the ultimate stress is inevitably much smaller than the ultimate stress found experimentally. Possibly the designers limited for composite structures the ultimate compressive stress of the fusées to a maximum equal to the allowable stress of concrete, then the ultimate stress for the fusées is equal to f_{fu} = 6 MPa. Nowadays the maximum values described in table 2.4 seem pretty low. At the present an ultimate value is defined as a minimal value, defined by subtracting the statistic uncertainly from the average value, so the ultimate stress f_u is equal to:

$$f_u = x_n - c\,s \qquad [2.17]$$

Where x_n is the average value, c is a parameter defined with the Theory of Probability and s is the standard deviation of the population wth n elements.
Further the design loads are defined by multiplying the representative loads with a load factor. In the past the stresses due to the representative loads were compared with an allowable stress. The allowable stress is calculated by diving an average value by a safety factor. In the present the classes of concrete are based on the compressive strength found by testing cylinders or cubes. Since 1974 [C9] minimal twelve cubes with an edge length of 150 mm were tested according to the code demanded in the Netherlands. Next the characteristic compressive strength is found with: $f_{ck,cube} = x_{12} - c\,s$
According to the code of 1950 the compressive strength of concrete was defined by compressing testing at least three cubes with the average compressive strength at least 25 MPa. The edge length of the cubes had to be 200 mm minimal. Between the load and the cube a thin plate of cardboard had to be laid to distribute the load over the area. Later the code of 1962, VBC 1962 [C2], specified three classification K160, K225 and K 300. The strength of concrete specified in GBV 1950 [C1] can be classified as K250 to conform the classification of the VBC 1962 [C2]. The classification of the code of 1962 was different from the classification of the codes of 1974/1984 [C9], but the classification of 1974/1984 does not differ much from the classification required in the Euro code [C6].
According to Van Boom [Boo77] the strength of cubes with a side of 150 mm without using a patch of cardboard is 16% larger than cubes with sides 200 mm. For the classes K160, K225 and K300, conform the code of 1962, Van Boom gives maximum values for the deviation equal to respectively s = 4.6 MPa, 6.1 MPa and 7.6 MPa according to the codes of 1974/1984 [C9]. These maximum values are based on a large number of experiments. For a large population the factor c is for a change of 5% equal to 1.64. Consequently the compressive strength can be calculated with:

$$f_{ck,cube} = 1.16\,x_{n=\infty} - 1.64\,s \qquad [2.17']$$

Table 2.5 shows the conversion of the classification of the VBC 1962 into the classification conform the Euro code 2 based on the strength of cubes.

Class	average value x_{mean} [MPa]	Deviation s [MPa]	Compressive strength of cubes $f_{ck,cube}$ [MPa]	Indication of the class conform the Eurocode
K160	16	4.6	1.16 × 16 – 1.64 × 4.6 = 11	C9/11
K225	22.5	6.1	1.16 × 22.5 – 1.64 × 6.1 = 16.1	C12/15
K300	30	7.6	1.16 × 30 – 1.64 × 7.6 = 22.3	C18/22

TABLE 2.5 Classes conform VBC 1962 [C2] and the Euro code [C6]

For K250 (according to GBV 1950 [C3]) the deviation s will be about 6.6 MPa, then the compressive characteristic strength of the cubes is equal to:

$$f_{ck,cube} = 1.16\, x_{12} - 1.64\, s = 1.16 \times 25 - 1.64 \times 6.6 = 18.1 \text{ MPa}$$

Thus the class K250 is approximately equal to the class given in the Eurocode [C9] C15/18. In 1950 for structures of concrete the safety factor concerning the loads was 1.8 and the factor concerning the deficiencies and variety of the strength of the material was 1.15 so the allowable stress had to be:

$$f_{cu} = 18/(1.8 \times 1.15) = 8.7 \text{ MPa}$$

So actually the ultimate compressive strength for bending, f_{cu} = 8 MPa, as described in GBV 1950, was quite reasonable compared with the maximum stress defined in the codes nowadays.

§ 2.3 The second order and slenderness according to the GBV 1950

For columns, subjected to normal forces, the stresses are increased due to second order effects. For these structures, subjected to a compressive normal force N, the buckling load N_{cr} must be significantly larger than the normal force N. According to Euler the buckling force is calculated with [2.12]:

$$N_{cr} = \pi^2 EI/l_c^{\,2} \qquad [2.12]$$

The ratio n_{cr} of the buckling force to the normal force is defined with [2.11]: $n_{cr} = N_{cr}/N$. The stress in the section of a structural element due to a normal force must be smaller than the ultimate stress. According to the code for structures of reinforced concrete of 1950 [C1], the stress due to a compressive force, is calculated by:

$$\sigma = N/A \le f'_c/\gamma \qquad [2.18]$$

So for a structure, subjected to a normal compressive force N, the stress is multiplied with a safety factor g to include the effect of the second order. In the code for structures of reinforced concrete of 1950 the safety factor γ was given in the following table with respect to the slenderness λ. The slenderness λ is defined with:

$$\lambda = l_c / i \qquad [2.19]$$

The factor i is the radius of gyration, this factor is equal to:

$$i = (I/A)^{1/2} \qquad [2.20]$$

A is the area of the section and I is the second moment of area. The slenderness can be calculated also by substituting the radius of gyration into the expression for the slenderness:

$$\lambda = l_c\, (A/I)^{1/2} \qquad [2.20']$$

The GBV 1950 [C1] prescribed for structures subjected to a normal load the safety factor γ with respect to a slenderness varying from 60 up to 140.

Slenderness λ	Safety factor γ
60	1.0
80	1.5
100	2.0
120	2.5
140	3.0

TABLE 2.6 The safety factor γ with respect to the slenderness λ according to the GBV 1950 [C1].

The reciprocal $1/\gamma$ shows the reduction of the normal stress with respect to the slenderness, see table 2.7. This factor $1/\gamma$ is approximately equal to the reduction factor ω that is used at the present for structures of steel or timber to describe the effect of buckling. For brittle structures such as concrete and masonry it is common practice to include the effect of buckling as an extra eccentricity of the normal load.
The reduction of the maximum stress to include the second order effects can be derived as follows. Firstly the slenderness [2.19], buckling force [2.12] and stress [2.18] are substituted into the expression for the ratio n_{cr} [2.11]:

$$n_{cr} = \frac{\pi^2 E}{\lambda^2 \sigma} \quad [2.21]$$

To avoid failure by buckling, the maximum stress in a section subjected to a normal compressive load is reduced with a ratio γ. Substituting $\sigma_u = f_{cu}/\gamma$ into [2.21] to find the ratio of n results in:

$$n_{cr} = \frac{\pi^2 E \gamma}{\lambda^2 f_{cu}} \quad [2.21']$$

With this expression for n_{cr} the buckling ratio is defined for several values of the slenderness.

Slenderness λ	Safety factor γ	Factor $\omega = 1/\gamma$	Buckling ratio n_{cr}	Ultimate stress $\sigma_u \leq f_{cu}\,\omega$.
60	1.0	1.0	9.6	6.0 MPa
80	1.5	0.67	8.1	4.0 MPa
100	2.0	0.50	6.9	3.0 MPa
120	2.5	0.40	6.0	2.4 MPa
140	3.0	0.33	5.3	2.0 MPa
160	3.5	0.29	4.7	1.7 MPa
180	4.0	0.25	4.3	1.5 MPa
200	4.5	0.22	3.9	1.3 MPa
220	5.0	0.20	3.6	1.2 MPa
240	5.5	0.18	3.3	1.1 MPa

TABLE 2.7 Slenderness, buckling ratio, stress and the factor ω

The Dutch code for structures of concrete of 1950 [C1] requires that the ultimate compressive stress in a structure subjected to a normal force was equal to: f_{cu} = 6.0 MPa. For centric loaded columns the ultimate stress was decreased to include the second order effects by dividing the ultimate stress with a factor γ: $\sigma < f_{cu}/\gamma = 6.0/\gamma$, γ is a safety factor depending on the slenderness λ of the column. The ratio γ is found in table 1 of the VBC 1950, according to article 3. Notice the factor γ is proportional to the slenderness λ.

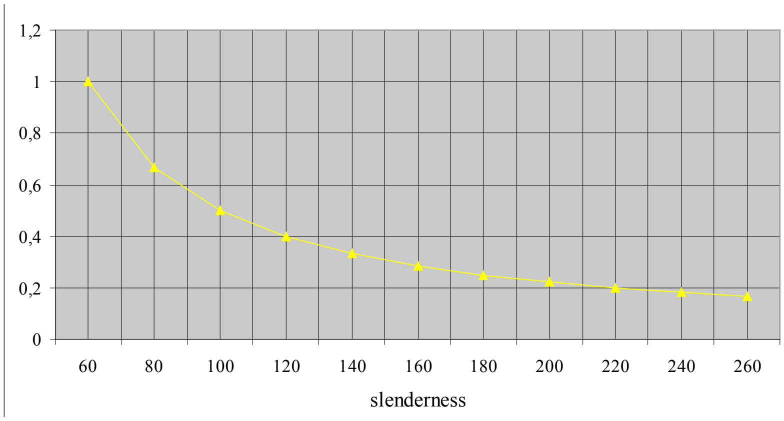

FIGURE 2.2 The ratio $\omega = 1/\gamma$ and the slenderness λ

Table 2.7 shows the results for a slenderness λ greater than 140 calculated by linear extrapolation of the values of the slenderness and γ. The factor n_{cr} is calculated with expression [2.11] for a concrete column with a Young's modulus of $E_c = 2.1 \times 10^4$ MPa and an ultimate stress of $f_{cu} = 6$ MPa. The values of γ shown in table 2.7 are identically to the values of γ shown in the VBC 1950 for $\lambda \leq 140$, see table 2.6. For $n_{cr} = 10$ the factor γ is equal to 1 and for $n_{cr} = 5.3$ $\gamma = 3.0$. In the VBC 1950 the factor γ was not given for a slenderness λ larger than 140. Probably the authors of the code of 1950 did not consider structural elements with a slenderness above 140 sufficiently safe.

§ 2.4 Validation of the calculation of the thrust for the two hinged vault

The vault is constructed on the building site and is spanning as a single element from support to support. The vault is supported by columns and edge beams. These supports cannot resist huge bending moments thus the supports are schemed as simple supports and the vault can be schemed as a two hinged arch. The magnitude of the thrust follows from the equations expressing the deformations of the arch due to the load and the thrust.

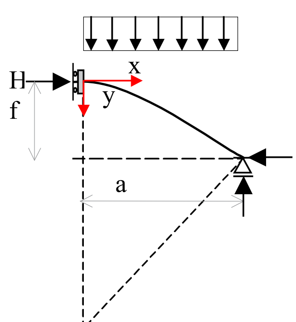

FIGURE 2.3 Half of a parabolic vault subjected to an equally distributed load q

The deformations of the curved element with a span $l = 2$ a and a rise f subjected to an equally distributed load q and a horizontal force H acting at the supports are calculated in chapter 6.

The horizontal deformations of the curved element, supposing the supports do not resist a horizontal forces follows from:

$$\Delta_q = \frac{4\,q\,f\,a^3}{15\,EI_o} \qquad [6.27]$$

The deformation of the curved element due to the force H acting on the supports is equal to:

$$\Delta_H = \frac{8\,H\,f^2\,a}{15\,EI_o} \qquad [6.28]$$

The deformation of the vault is reduced by the tie between the supports. The deformation of the tie with length a is equal to:

$$\Delta_T = \frac{H\,a}{EA_s} \qquad [2.22]$$

The deformation of the supports follows from: $\Delta_T = \Delta_{bq} - \Delta_H$. Substituting the deformations of the vault into this expression leads to:

$$\frac{H\,a}{EA_s} = \frac{4\,q\,f\,a^3}{15\,EI_o} - \frac{8\,H\,f^2\,a}{15\,EI_o} \qquad [2.23]$$

$$H = \frac{\tfrac{1}{2}\,q\,a^2}{f\,(1 + C)} \quad \text{with: } C = \frac{15\,EI_o}{8\,EA_s\,f^2} \qquad [2.24]$$

If the steel tie is very stiff, then the factor C is very small and the thrust is equal to:

$$H = {}^{1}/_{8}\,q\,l^2/f = \tfrac{1}{2}\,q\,a^2/f$$

The steel tie is selected according to the load and the maximum stress. According to the TBV 1955 the stress had to be less than σ_s = 140 MPa. The vault is subjected to a load q = 4.3 kN/m, the span is equal to 24 m so a = ½ *l* = 12.0 m and the rise f is equal to 3 m. For a tie Ø36 mm, with an area of A_s = 1017 mm^2, subjected to a thrust equal to H = 103.2 kN the stress is equal to: σ = 103200/1017 = 101 MPa. Further the stiffness of the arch is equal to EI = 12 × 10^{12} Nmm^2. Substituting a = 12 m, f = 3 m, q = 4.3 kN/m and EI = 12 × 10^{12} Nmm^2 in the expression to determine the thrust H gives:

$$H = 0.99 \times (\tfrac{1}{2}\,q\,a^2/f)$$

Due to the elongation of the tie the thrust is about 1% smaller than calculated before. Thus the reduction of the thrust due to the deformation of the tie is very small. It is quite understandable that Van Eck and Bish neglected the deformation of the tie.

§ 2.5 Approach of the surface load

The bending moment due to the surface load can be calculated easily by simplifying the surface load into a linear increasing load which is equal to q at the top and maximum at the footing q_{max} = q (1 + c). To simplify the calculation the assumption is made that the structure is hinged at the top. Then the structure is statically determinate. The effect of this assumption will be discussed in chapter 6. Due to the equally distributed load q the structure is subjected to a normal force. Due to the triangular load c q the structure is subjected to a normal force and a bending moment. The forces and moments are defined in chapter 6.

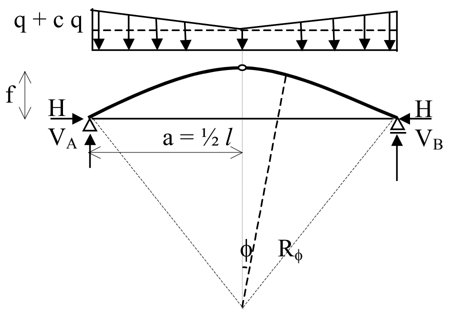

FIGURE 2.4 Parabolic vault subjected to an increasing load q + c q.

At a distance x from the top the load is equal to: $q_x = q\ ds/dx$. Substituting $ds = dx\ (1 + (dy^2/dx^2)^{0.5}$ gives for this load: $q_x = q\ (1 + dy^2/dx^2)^{0.5}$.

Substituting $dy/dx = 2\ f\ x/a^2$ gives: $q_x = q\ (1 + 4\ f^2\ x^2/a^4)^{0.5}$

For x = a the load is at maximum: $q_{x=a} = q\ (1 + 4\ f^2/a^2)^{0.5}$

The factor c follows for x = a from: $q\ (1 + c) = q\ (1 + 4\ f^2/a^2)^{0.5}$ →

$$c = (1+ 4\ f^2/\ a^2)^{½} \qquad [2.25]$$

The vertical reaction V acting at the supports is equal to: $V = q\ a + ½\ c\ q\ a$ [2.26]

The thrust H follows from the equilibrium of bending moments around the top:

$$H\ f - V\ a + ½\ q\ a^2 + {}^1/_3\ c\ q\ a^2 = 0$$

Substituting V: $H = (½\ q\ a^2 + {}^1/_6\ c\ q\ a^2)/\ f$ → $H = ½\ (1+ {}^1/_3\ c)\ q\ a^2/f$ [2.27]

The normal force acting at the vault is maximum equal to: $N = (H^2 + V^2)^{½}$

Substituting H and V: $N = q\ a\ \{\ [\ (1+{}^1/_3\ c)\ ½a/f]^2 + (1+½\ c)^2\ \}^{½}$

The bending moment is at maximum for $x = {}^2/_3\ a$ from the top, the maximum bending moment follows from [6.18]:

$$M_x = {}^2/_{81}\ c\ q\ a^2 \qquad [6.18]$$

The vault is subjected to the dead load $q_g = 3.3\ kN/m^2$; for a = 12 m and f = 3 m we find:

$c = (1+ 4 \times 3^2/12^2)^{½} - 1= 0.118$

$V = q\ a\ (1 + ½\ c) = 3.3 \times 12 \times (1+0.118/2) = 41.94\ kN$

$H = ½\ q\ (1+ {}^1/_3\ c)\ a^2/f = ½ \times 3.3 \times (1+ {}^1/_3\ 0.118) \times 12^2/\ 3 = 82.3\ kN$

$N = (H^2 + V^2)^{½} = 92.4\ kN$

$M_x = {}^2/_{81} \times 0.118 \times 3.3 \times 12^2 = 1.4\ kNm$

The vault is not subjected to bending to the dead load if the shape of the vault follows a catenary. The differences between a parabola and catenary are very small.

The coordinates of the parabola follows from [1.1]: $y/f = (x/a)^2$

The coordinates of a catenary follows from [1.4]: $y/f = [\cosh (x/c) - 1]\ c/f$ With $c = H/q$

To describe the catenary for a given span and rise Te Boveldt [Bov94] gives an expression to approach the coefficient c with:

$$2a/c = \frac{4\,f/a}{(1+ \tfrac{1}{2}\, f^2/a^2)^{1/2}} \qquad [2.27]$$

In practice most vaults were designed with a ratio rise to span f/a = ¼ , then c is equal to 2.031 a. Probably in the past most engineers, using a sliding rule, designed the catenary with c = 2 a.
An accurate approach is found with c = 2.042 a, then the coordinates of a catenary with f/a= ¼ follows from:

$$y/a = 2.042 \times [\cosh(0.4897\, x/a) - 1] \qquad [2.28]$$

The foll owing table shows the coordinates for the parabola and the catenary. The differences between the parabola and catenary are very small for a curve with a small ratio rise to span f/a. For the curve with f/a = ¼ the maximum difference is $\Delta/f = 0.005$.

x/a	Parabola y/a=	Catenary y/a =	Difference Δ/a =
0	0	0	0
0.1	0.0025	0.00245	0.00005
0.2	0.0100	0,00980	0.00020
0.3	0.0225	0,02208	0.00042
0.4	0.0400	0.03930	0.00070
0.5	0.0625	0.06152	0.00098
0.6	0.0900	0.08879	0.00122
0.7	0.1225	0.12116	0.00134
0.8	0.1600	0.15872	0.00128
0.9	0.2025	0.20157	0.00093
1	0.2500	0.24979	0.00021

TABLE 2.8 Coordinates for a parabola and the catenary for f/a = ¼

Bending moment due to surface load

For a parabolic vault the bending moment due to the surface load is defined. The normal force follows the line of the system (the catenary), the bending moment due to the dead load is equal to: M = N e, where e is the eccentricity following from: **$e/a = (\Delta/a)\cos\phi$**

For x/a = 0.7 the difference (Δ/a) is equal to 0.00134. For x/a = 0.7 the angle ϕ follows from $\tan\phi$ = $dy/dx = 2\,f\,x/a^2 = 0.35$, thus $\cos\phi = 0.944$.
Then the eccentricity is: $e = 0.00134 \times 12.0 \times 0.944 = 0.0152$ m

The bending moment due to the dead load is equal to: $M = N\,e = 92.4 \times 0.0152 = 1.4$ kNm.

Due to the increasing load the vault is subjected to bending moments. The bending moments due to this load are minor, significantly less than the bending moments resulting from an asymmetric load.

§ 2.6 Verification of the scheme and assumptions concerning the permanent load

The vault was schematized as an arch supported with a roller and a hinge which are joined with a tie. The force in the tie is calculated for the arch schemed as a statically determinate structure. Actually the vault is statically indeterminate since the deformation of the tie can affect the magnitude of the force in the tie. If the force in the tie is larger or smaller than the force calculated for the statically determinate structure, then this structure is subjected to bending moments even if the structure is only subjected to a symmetrical load. Furthermore the dead load is an equally distributed surface load. For a symmetrical equally distributed load a parabolic arch will be subjected to normal forces only, but a parabolic arch subjected to an equally distributed surface load will also be subjected to bending moments. Nowadays the forces and bending moments in an arch can be calculated easily with a finite element program.

The vault is subjected to an equally distributed surface load. At the centre the load is equal to $q_x = 3.3$ kN/m. At a distance x the load will be larger than the load at the centre, $q_x = 3.3 \times ds/dx$ with:

$$ds = dx\,(1 + dy^2/dx^2)^{½}.$$

Thus the load q_x is calculated with:

$$q_x = 3.3 \times [1 + (dy/dx)^2]^{½}$$

The parabolic vault is described with the centre of the coordinates at the top with [1.1]:

$$y = f\,x^2/a^2 \quad [1.1]$$

Differentiation gives the tangent:

$$y' = 2\,f\,x/a^2 \quad [2.29]$$

The length of a small part of the curve ds is defined with:

$$ds = dx\,(1 + y'^2)^{1/2} \quad [2.30]$$

Substituting $y' = 2\,f\,x/a^2$ into the expression for ds:

$$ds = dx\,(1 + 4 \times f^2 x^2/a^4)^{1/2} \quad [2.31]$$

For the parabolic vault with a rise f = 3.0 m and a span *l* = 24.0 m the ratio ds/dx is equal to:

$$ds/dx = (1 + 4 \times 3.0^2\,x^2/12^4)^{1/2} = (1 + 0.001736\,x^2)^{1/2}$$

For dx = 1.0 m the dead load at a certain point x is equal to:

$$q_x = 3.3 \times (1 + 0.001736\,x^2)^{1/2}$$

The following table shows the dead load for the vault with a width of 1.0 m for dx = 1.0 m , the coordinates of nodes, the position of the members and the results of the calculations.

The dead load is increasing from the centre to the supports. Due to this dead load the vault is subjected to bending moments. These bending moments are minor, significantly smaller than the bending moments due to the asymmetrical live load. Due to the asymmetric live load the bending moments are identically to the bending moments calculated before. Actually the calculations made manually with a slide rule sixty years ago are surprisingly accurate.

x =	y =	dead load q_G	Node	x =	y =	Member
0	0	3.3	1	0	0	S1: 1- 2
1	0.021	3.303	2	2	0.92	S2: 2- 3
2	0.083	3.31	3	4	1.67	S3: 3- 4
3	0.188	3.326	4	6	2.25	S4: 4- 5
4	0.333	3.346	5	8	2.67	S5: 5- 6
5	0.521	3.371	6	10	2.92	S6: 6- 7
6	0.75	3.402	7	12	3.0	S8: 7 - 8
7	1.021	3.438	8	14	2.92	S9: 8- 9
8	1.33	3.479	9	16	2.67	S10: 9-10
9	1.688	3.524	10	18	2.25	S11:10-11
10	2.083	3.575	11	20	1.67	S12:11-12
11	2.521	3.63	12	22	0.92	S13:12-13
12	3.0	3.69/2	13	24	0	S14: 1-13

Member	dead load	dead load	Live load sym.	live load sym.	live load asym	live load asym
	N =	M =	N =	M =	N =	M =
S1	91.2	2.0	26.9	0	12.2	- 5.1
S3	85.7	1.5	25.4	0	12.4	- 9.1
S6	81.6	1.7	24.2	0	12.2	0
S10	85.7	1.5	25.4	0	13.0	8.9
S12	91.2	1.0	26.9	0	14.7	4.9
S13	81.4	0	24.1	0	12.1	0

TABLE 2.9 Normal forces and bending moments due to the dead load and the live load acting symmetrical and asymmetrical

§ 2.7 Ultimate load bearing capacity

Nowadays the reinforcement has to be calculated according to Eurocode 2 [C6]. The calculation of the required reinforcement is based on a non-linear stress-strain diagram of the concrete and steel, the stress-strain diagrams of concrete and steel are simplified and schematized using a rectangular approximation. Due to the infill elements the section is not massive. The cover on the hollow core is c_f. For a compressive zone k_x h larger than c_f the compressive zone has to be reduced. For this structure with two layers of fusée elements the normal compressive component is reduced with a force F_f acting opposite the reaction force due to the concrete reaction force F_c acting on the compressive zone x = k_x h.

Features of steel

The stress in the steel reinforcement must be less than the ultimate stress $\sigma_s < f_{yd}$. The stress-strain diagram of the steel is bi-linear: for $\varepsilon_s < f_{yd}/E_s$ the maximum stress follows from $\sigma_s = \varepsilon_s E_s$ and for $f_{yd}/E_s < \varepsilon_s < \varepsilon_{su}$ the maximum stress is equal to f_{yd}. The design load follows from: $f_{yd} = f_{yk}/\gamma_s$ with: $\gamma_s = 1.15$. For FeB220: $f_{yd} = 220/1.15 = 191$ MPa, $E_s = 2 \times 10^5$ MPa.

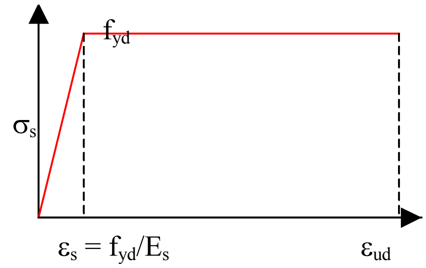

FIGURE 2.5 The bi-linear stress-strain of the steel rebars

Features of concrete

The quality of concrete is described with two numbers, the first one mentions the cylindrical strength, the second one the strength of cubes. For C12/15 the cylindrical strength is equal to 12 MPa and the strength of the cubes is equal to 15 MPa. The compressive stress in the concrete must be less than the ultimate stress: $\sigma_c < f_{cd}$, with $f_{cd} = f_{ck}/\gamma_c$; f_{ck} is the cylindrical strength, generally the safety factor is equal to: $\gamma_c = 1.5$. Thus for C12/15 the ultimate stress is equal to $f_{cd} = f_{ck}/\gamma_c = 12/1.5 = 8$ MPa.

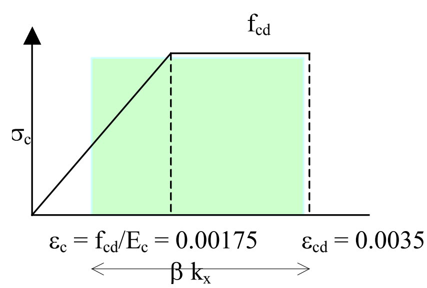

FIGURE 2.6 The bi-linear stress-strain of the concrete

Generally for concrete the stress-strain diagram is schemed parabolic, bi-lineair or rectangular. For the bilinear stress-strain diagram the strain is at maximum $\varepsilon_{cu} = 0.0035$. The ultimate stress increases linear for $\varepsilon_{cu} \leq 0.0035/2$. The ultimate stress is constant for $0.00175 < \varepsilon_{cu} \leq 0.0035$. The bi-linear stress-strain diagram includes implicite the effect of the time dependent deformation due to creep. For the ultimate state the stiffness of the concrete follows from: $E_{cd} = f_{cd} /(0.5\ \varepsilon_{cu})$. Thus for C12/15 Young's modulus, including creep, is for the ultimate state equal to $E_{cd} = 8/(0.00175) = 4571$ MPa.

At the present for the ultimate load bearing capacity the stress-strain diagram can be schematized rectangular with $\sigma = f_{cd}$ and a reduction factor for the depth $\beta = 0.8$ [C6]. The compressive zone is equal to $x = k_x h$. The normal compressive component F_c acting at the compressed side of the concrete follows from: $F_c = \beta\, k_x\, h\, f_{cd}$

Due to the fusées the normal compressive component is reduced with a force F_f following from:

$$F_f = m\, A_f\, f_{cd}$$

where: m is the number of fusées

A_f is the area of the fusées within the compressive zone $\beta\, k_x\, h$

The specific deformation of the steel at the tensioned and compressed side of the section follows respectively from:

$$\varepsilon_{st} = (1 - d/h - k_x)/k_x \quad \text{and} \quad \varepsilon_{sc} = (k_x - d/h)/k_x$$

With: d = distance from the centre of reinforcement to the nearest side.

For $\sigma_s < f_{yd}/E_s$ the stress in the steel reinforcement follows from: $\sigma_s = \varepsilon_s E_s$;
For $\sigma_s > f_{yd}/E_s$ the stress in the steel reinforcement is equal to f_{yd}: $\sigma_s = f_{yd}$;
f_{yd} is the design load with: $f_{yd} = f_{yk}/\gamma_s$ and $\gamma_s = 1.15$.

The sections of the vault are subjected to a normal force N_d acting eccentrically at a distance e_t from the centre. The reinforcement is placed symmetrically in the section, with $e_t = M_d / N_d$.
The ultimate normal force and bending moment follows from respectively:

$$N_d = F_c - \Sigma F_f + F_{sc} - F_{st} \qquad [2.32]$$

$$M_d = (F_{sc} + F_{sc}) \times (½\, h - d) + F_c\,(½\, h - \beta_1) - F_{f1}\,(½\, h - z_1) + F_{f1}\,(½\, h - z_2) \qquad [2.33]$$

Where: $F_{st} = F_{sc} = ½\, A_s\, \sigma_s$

For the vault with two rows the effect of the fusées situated at the compressed side is: $F_{f1} = A_{f1}\, \sigma_c$

For $k_x \leq 1.25\, c_f/h$: $A_{f1} = 0$; $z_1 = 0$
For $k_x = 1.25 \times (c_f/h + r/h)$ $A_{f1} = \pi\, r^2/2$; $z_1 = t + (1\text{-}0.424)\, r$
For $k_x \geq 1.25 \times (c_f/h + 2\, r/h)$ $A_{f1} = \pi\, r^2$; $z_1 = t + r$

For the vault with two rows the effect of the fusées situated at the tensioned side is: $F_{f2} = A_{f2}\, \sigma_c$

For $k_x \leq 1.25 \times (1 - c_f/h - 2\, r/h)$: $A_{f2} = 0$
For $k_x = 1.25 \times (1 - c_f/h - r/h)$ $A_{f2} = \pi\, r^2/2$ $z_1 = t + (1\text{-}0.424)\, r$
For $k_x \geq 1.25 \times (1 - c_f/h - 2\, r/h)$ $A_{f2} = \pi\, r^2$ $z_1 = t + r$

Where: $t = (h - 4\, r - 2\, c_f)/2$

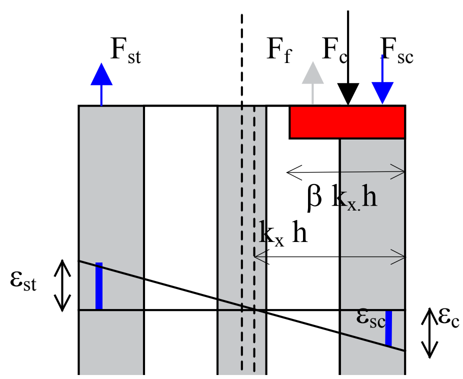

FIGURE 2.7 Forces, stresses and specific deformations for an eccentric loaded section with two rows of fusées

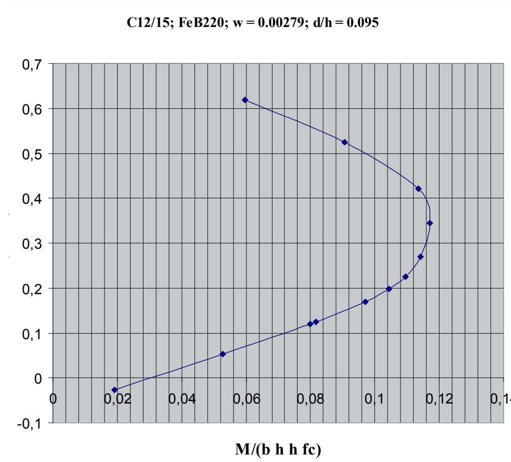

FIGURE 2.8 Graph showing the ultimate bearing capacity of the vault

The calculation of a column is quite labour intensive, most engineers will use diagrams or spreadsheets to calculate the load bearing capacity of a section subjected to an eccentric normal force.

The graph, see figure 2.8, illustrates the load bearing capacity for a vault with C12/15; Fe220; d/h = 0.1 and a reinforcement $\omega = A_t/(bh) = 2 \times 279/200000 = 0.0028$. The graph shows on the vertical axis $N_d/(b\, h\, f_{cd})$ and on the horizontal axis $M_d/(b\, h^2 f_{cd})$. For a section subjected to a normal force the ultimate bending moment can be calculated also with the following table

$N_d/(bh\, f_{cd})$	$M_d/(b\, h\, h\, f_{cd})$
-0.0268	0.0192
0.0527	0.0528
0.1200	0.0798
0.1250	0.0817
0.1696	0.0970
0.1973	0.1043
0.2249	0.1094
0.2695	0.1143
0.3445	0.1171
0.4212	0.1135
0.5253	0.0907
0.6185	0.0597

TABLE 2.10 Bearing capacity for a vault with two layers of fusées C12/15; Fe220; d/h = 0.1; $\omega = 0.0028$

For the ultimate state the permanent and live load are increased with a load factor of respectively 1.2 and 1.5.The permanent load is equal to $q_g = 1.2 \times 3.3$ kN/m. The live load is equal to $q_e = 1.5 \times 1.0$ kN/m. The normal force for the permanent and asymmetrical load and the bending moment is equal to:

$$N_d = 1.2 \times 85.7 + 1.5 \times 12.4 = 121.4 \text{ kN}$$

$$M_o = 1.5 \times 9.1 = 13.7 \text{ kNm}$$

With $f_{cd} = 12/1.5 = 8$ MPa the ratio for the normal force is: $\dfrac{N_d}{b\, h\, f_{cd}} = 0.076$

Table 2.10 shows that for this normal force the maximum bending moment is equal to:

$$\frac{M_d}{b\, h^2 f_{cd}} = 0.062$$

Then the maximum bending moment the section can resist is: M = 19.8 kNm. To include the second order effects the ratio n has to be defined again for the ultimate state.

§ 2.8 Ultimate state: stiffness and buckling

The stiffness is affected by cracks and creep of the concrete. According to NEN-EN 1992-1-1 [C6] the stiffness can be approached for the ultimate state with:

$$EI = K\, E_{cd} I_0 + E_s I_s$$

With: $K = \frac{k_1 k_2}{1 + \phi_{ef}}$ $k_1 = \sqrt{(f_{ck}/20)}$ $k_2 = \frac{N_d \lambda}{A_c f_{cd} 170}$ $\phi_{ef} = \phi_t M_{Ef}/M_{Ed}$

Creep

According to the NEN-EN 1992-1-1 [C6] the specific creep is calculated. The moulds were reused as fast as possible, but the time before the mould could be removed had to be at least 36 hours and the strength of the concrete had to be developed enough at this stage. To allow for this probably cement class N was used and the time t_0 was at least 2 days. Generally warehouses were slightly heated, the RH was approximately 70%.
According to NEN-EN 1992-1-1, figure 3.1:

For RH = 50% the specific creep is: $\phi(\infty,t_0) = 4.8$
For RH = 80% the specific creep is: $\phi(\infty,t_0) = 3.6$

For RH = 70% the specific creep is: $\phi(\infty,t_0) = 3.6 + \frac{(80 - 70) \times (4.8-3.6)}{(80 - 50)} = 4.0$

For a concrete structure subjected to a permanent compressive load the instantaneous specific deformation is ε. Due to creep the specific deformation will increase with $\phi\ \varepsilon$. The total deformation of the concrete is equal to $\varepsilon\ (1+\phi)$ with $\phi = 4.0$.
According to the Eurocode the effective creep factor ϕ_{ef} follows from:

$\phi_{ef} = \phi_t M_{Ef}/M_{Ed}$

Due to the permanent load the vault is subjected to a bending moment:

$M_{E\phi} = {}^2/_{81} \times 0.118 \times q\, a^2 = {}^2/_{81} \times 0.118 \times 3.3 \times 12^2 = 1.4$ kNm

The maximum moment due to the asymmetrical load is equal to:

$M_{Ed} = 1.5 \times 9 = 13.5$ kNm

The effective creep factor ϕ_{ef} is equal to: $\phi_{ef} = \phi_t\, M_{rep}/M_{Ed} = 4.0 \times 1.4/13.5 = 0.41$

For the instantaneous load Young's modulus follows from:

$E_t = E_{cd} /(1+ \phi_{ef})$ $\rightarrow$ $E_t = 22500/(1+0.41) = 15957$ MPa

The stiffness is according to Euro code [C6] calculated with: $EI = K\, E_{cd} I_0 + E_s I_s$

$E_s I_s = 200000 \times ½ \times 558 \times 2 \times (100-19)^2 = 0.73 \times 10^{12}$ Nmm²

$K = k_1 k_2 /(1+ \phi_{ef})$

Where: $k_1 = \sqrt{(f_{ck}/20)} = (12/20)^{1/2} = 0.775$ and $k_2 = N_d \lambda /(A_c f_{cd} 170)$

The slenderness follows from [2.19]: $\lambda = l_c / i$ where $l_c = s = 12.5$ m and $i = \sqrt{I/A}$

$I_o = 1000 \times 200^3/12 - 2 \times 11 \times \pi\, 60^2/4 \times 45^2 = 5.4 \times 10^8$ mm⁴
$A_c = 1000 \times 200 - 2 \times 11 \times \pi\, 60^2/4 = 137.8 \times 10^3$ mm²
$i = \sqrt{I/A} = 63$ mm

Substituting s and i gives $\lambda = 198$

$$k_2 = \frac{N_d \lambda}{A_c f_{cd} 170} = \frac{121400 \times 198}{137800 \times 8 \times 170} = 0.128$$

$$K = \frac{k_1 k_2}{1 + \phi_{ef}} = \frac{0.775 \times 0.128}{1 + 0.5} = 0.066$$

$$EI = 0.066 \times 15957 \times 5.4 \times 10^8 + 0.73 \times 10^{12} = 1.3 \times 10^{12}\ Nmm^2$$

The stiffness described by Bish and Van Eck was equal to $EI = 12 \times 10^{12}$ Nmm^2. Due to time-dependent effects and cracking the stiffness is decreased substantially. Next the critical buckling force is calculated with equation [2.6]. Substituting f = 3.0 m; R = 25.6 m, $EI = 1.3 \times 10^{12}$ Nmm^2; $\phi = 0.49$ radians in [2.6] gives::

$$N_{cr} = \frac{EI\,[\pi^2/\phi^2 - 1]}{R^2} = \frac{1.3 \times 10^{12} \times [\pi^2/0.49^2 - 1]}{25600^2} = 79.6 \times 10^3\ N$$

The design load N_d is equal to 121.4 kN so this force is larger than the buckling load N_{cr}. The structure cannot resist the loads including the load factors.

§ 2.9 Conclusions

The analysis shows that in the article written sixty years ago the effect of the time dependent deformations was underestimated. Due to the decrease of the stiffness the critical buckling force is decreased too. The critical buckling force is smaller than the normal force due to the design loads. This structure is not safe regarding the buckling load and has to be strengthened to meet the demands of the present.

3 In-plane buckling of arches and vaults

Arches and vaults subjected to a transversal load can collapse due to in-plane buckling. Arches with a very small rise can change form from convex to concave. The structure will snap through . The deformation of an arch failing by snapping through is symmetrical. The deformation of an arch failing by buckling can be also asymmetrical. Vandepitte [4] showed that for a sinusoidal arch subjected to a sinusoidal load the asymmetrical buckling mode is more critical than the symmetrical snap through mode. Generally, even for shallow arches, the asymmetric deformation induces the critical buckling load. In the past the critical buckling load was only defined for circular arches and vaults. Generally the Fusée Céramique vaults were designed with a parabolic curvature. Fortunately for shallow arches the differences between a parabola and a circular curve are minor. Nevertheless this chapter will study this problem. Usually the effect of the stiffness of supports on the critical buckling load is neglected. A tie joining the supports of an arch or vault will elongate and thus decrease the critical buckling load. Possibly neglecting the stiffness of the supports can overestimate the critical buckling load and cause failure by snap through. The effect of the flexible supports concerning the buckling load will be studied too. This chapter compares various theories to find the buckling load for shallow parabolic vaults with a ratio of rise to span of about $f = {}^{1}/_{8}\, l$. First the theories developed in the fifties describing the buckling force for arches deforming asymmetrically are looked at. Next more recent research to define the buckling load for the parabolic and circular vaults more precisely is taken into account. Finally the effect of the vertical hangers between arch and tie is considered.

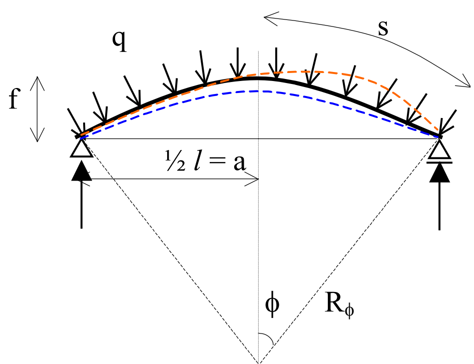

FIGURE 3.1 Two hinged vault ,subjected to radial load, symmetric and asymmetric deformation

§ 3.1 Buckling

As shown in chapter two, Bish and van Eck defined for a Fusée Céramique vault the critical buckling load with the following formula defined by Timoshenko [Tim52] for the asymmetrical mode of a circular two hinged arch, subjected to a radial load, see expression [2.6]:

$$N_{cr} = q_{cr} R = \frac{EI\,(\pi^2/\phi^2 - 1)}{R^2} \qquad [2.6]$$

Where ϕ is the angle describing the arch from the top to the support. The length of the circular arch between the supports is equal to 2 s. With s = R ϕ the critical buckling for an asymmetrical de formation is equal to:

$$N_{cr} = \frac{\pi^2 EI (1 - \phi^2/\pi^2)}{s^2} \quad [3.1]$$

According to Euler the buckling force is equal to:

$$N_{cr} = \frac{\pi^2 EI}{l_c^2} \quad [3.2]$$

The buckling length of an arch is equal to $l_c = \psi$ s. Substituting l_c into [3.2] gives:

$$N_{cr} = \frac{\pi^2 EI}{(\psi s)^2} \quad [3.3]$$

Comparing expression [3.1] and [3.3] shows that the factor ψ, defining the buckling length with respect to the length of the arch, has to be equal to:

$$\psi^2 = 1/(1 - \phi^2/\pi^2) \quad [3.4]$$

For $\phi < \pi$ the factor ψ^2 is larger than 1, consequently $\psi > 1$. The ratio of the rise and span of an arch is related to the angle ϕ. For a circular arch the span and rise are respectively equal to $l = 2a = 2R \sin\phi$ and $f = R(1 - \cos\phi)$, thus the ratio f/a follows from:

$$f/a = (1 - \cos\phi)/\sin\phi \quad [3.5]$$

Table 3.1 shows the ratio factor ψ for several values of ϕ and the ratio f/a.

Angle ϕ	Angle ϕ [radians]	$[1 - \phi^2/\pi^2]$	ψ =	Ratio f/a =	Ratio f/l
90°	½ π	0.750	1.154	1.0	0.500
75°	5 π /12	0.826	1.100	0.767	0.384
60°	π/3	0.889	1.061	0.577	0.289
45°	π/4	0.938	1.032	0.414	0.207
30°	π/6	0.972	1.014	0.268	0.134
15°	π/12	0.993	1.004	0.132	0.066

TABLE 3.1 The buckling length of a two hinged circular arch or vault

For shallow curved arches with $\phi < 30°$ the buckling length is nearly equal to the length of the arch s between the support and the top. Then the buckling force can be approached using the expression of Euler with l_c = s. In the same period, halfway the twentieth century, Goldenblat and Sisow [Gol55] derived the critical buckling load q_{cr} for circular tubes, subjected to an equally distributed radial load:

$$q_{cr} = \frac{E(t/R)^3}{4(1 - \upsilon^2)} \quad [3.6]$$

Where υ = the coefficient of lateral contraction.

For a section with a thickness t and width b the second moment of the area is equal to: $I = b t^3/12$. Substituting the second moment of the area I into [3.6] gives for $\upsilon = 0$ a critical buckling load q_{cr} equal to:

$$q_{cr} = \frac{3 EI}{R^3} \quad [3.6']$$

To including the effect of varying supports, Goldenblat and Sisow [Gol55] defined for circular arches, subjected to a radial load, the following expression:

$$N_{cr} = \frac{k\,EI}{R^2} \qquad [3.7]$$

The factor k is a variable dependent on the boundary conditions. The following table shows k for two hinged arches, three hinged arches and arches with clamped supports.

	Clamped supports k =	Two hinges k =	Three hinges k =	f/a =	f/(2 a)
ϕ = 90°	8	3	3	1.0	0.5
ϕ = 75°	11.5	4.76	4.32	0.767	0.384
ϕ = 60°	19.1	8	6.75	0.577	0.289
ϕ = 45°	32.4	15	12	0.414	0.207
ϕ = 30°	74.3	35	27.6	0.268	0.134
ϕ = 15°	294	143	108	0.132	0.066

TABLE 3.2 The factor k for a varying angle ϕ and varying boundary conditions

The expression of the buckling force according to Goldenblat and Sisow [3.7] can be compared with the expression of Euler [3.3] with a buckling length $l_c = \psi\, s$, and $s = \phi\, R$. Then the factor ψ follows from:

$$\frac{k\,EI}{R^2} = \frac{\pi^2\,EI}{(\psi\,\phi\,R)^2} \quad \rightarrow \quad \psi^2 = \frac{(\pi/\phi)^2}{k} \qquad [3.8]$$

Table 3.3 shows the factor ψ for circular arches subjected to a radial load with respect to the angle ϕ.

	Two hinged arch Ψ =	Three hinged arch Ψ =	f/a	f/l
ϕ = 90°	1.155	1.155	1	0.5
ϕ = 75°	1.100	1.155	0.767	0.384
ϕ = 60°	1.061	1.155	0.577	0.289
ϕ = 45°	1.033	1.155	0.414	0.207
ϕ = 30°	1.014	1.155	0.268	0.134
ϕ = 15°	1.004	1.155	0.132	0.066

TABLE 3.3 The ratio ψ with respect to the rise f and span a

Comparing table 3.1 and table 3.3 shows that for the two-hinged arch the buckling length according to Timoshenko does not vary much from the buckling length according to Goldenblat et al.

Parabolic arch

For a shallow arch the curvature of a parabola does not vary significantly from an arch following a circle. Thus for parabolic low rise arches the expressions defined for circular arches are used often to define the buckling load. For deep arches the difference increases. Te Boveldt [Bov94] gives for parabolic arches the following table for the buckling length: $l_c = \psi\, s$. As described in chapter 2 the length s can be calculated for a parabola with:

$$s = f\,(1+ \tfrac{1}{4}\, a^2/f^2)^{1/2} + \tfrac{1}{4}\,(a^2/f)\ \ln\{2\,f/a + (4\,f^2/a^2+1)^{1/2}\} \quad [2.13]$$

Comparing table 3.2 and table 3.3 with table 3.4 shows for parabolic arches, especially if the rise increases, a larger buckling length than for a circular arch. For shallow arches with a rise to span ratio $f/l < 0.2$ the difference is small.

		$f/l = 0.05$	$f/l = 0.2$	$f/l = 0.3$	$f/l = 0.4$	$f/l = 0.5$
Three hinges	$\psi =$	1.2	1.16	1.13	1.19	1.25
Two hinges	$\psi =$	1.0	1.06	1.13	1.19	1.25
Clamped supports	$\psi =$	0.7	0.72	0.74	0.75	0.75

TABLE 3.4 The buckling length of parabolic arches $l_c = \psi.s$ with respect to the rise f and span l

§ 3.2 Non-linear Analysis

Recently Pi et al [PiY02] and Moon et al [Moo07] researched the critical buckling load for respectively circular and parabolic shallow pin-ended arches, supposing unmovable supports.

Circular pin-ended arches

Recently Pi et al [PiY02] defined for circular shallow arches a non-linear in-plane analysis. For pin-ended arches the critical in-plane symmetric buckling load causing snap through is equal to:

$$N_{cr\,sym} = \frac{\pi^2 EI}{(2\,s)^2} \quad [3.9]$$

Where (2 s) is the length of the arch between the supports.

According to Pi the in-plane asymmetric buckling load follows from:

$$N_{cr\,asym} = [\,0.26 \pm 0.74 \times (1- 0.63\,\pi^4/\lambda^2)^{1/2}\,]\ \frac{\pi^2 EI}{s^2} \quad [3.10]$$

With the slenderness $\lambda = s^2/(R\,i)$ and the radius of gyration of the section $i = (I/A)^{1/2}$

According to Pi the asymmetrical buckling mode is more critical in case: $\lambda > 9.38$
Snap through is more critical in case: $\lambda < 7.83$

For circular arches the length s is equal to $R\,\phi$ and the span is equal to $l = 2\,a$, with $a = R \sin\phi$.
Substituting the radius $R = a/\sin\phi$ into $s = R\,\phi$ gives: $s = a\,\phi/\sin\phi$.

Substitute $s = R\,\phi$ and $s = a\,\phi/\sin\phi$ into the condition for the slenderness: $\lambda > 9.38$, shows the asymmetrical buckling mode is more critical for:

$$\frac{R\,\phi\,(a\,\phi/\sin\phi)}{R\,i} > 9.38 \quad \rightarrow \quad i/a < \frac{0.1066\,\phi^2}{\sin\phi} \quad [3.11]$$

For an arch or vault with a rectangular section and depth t the radius of the section is equal to i = 0.289 t. Substituting the radius of the section into [3.11] gives:

$$t/a < 0.369\, \phi^2/\sin\phi \qquad [3.11']$$

Generally the Fusée Céramique vaults were designed with a ratio f/a = ¼. Substituting tan ϕ = 0.25 and ϕ = 0.245 radians into [3.11'] shows the asymmetrical buckling load is critical for:

$$t/a < 0.09.$$

The Fusée Céramique vaults were designed quite slender, with a thickness of 110 mm for a span up to 14.4 m. For these vaults the switch is equal to: t/a = 110/7200 = 0.015 < 0.09, so the asymmetrical buckling mode is critical.

Parabolic pin-ended arches.

Moon et al [Moo07] researched the critical buckling load for parabolic pin-ended arches. For in-plane asymmetric buckling mode the critical buckling load is equal to:

$$N_{cr\,asym} = \frac{\pi^2 EI}{(2\,\psi'\,s)^2} \qquad [3.12]$$

For pin-ended arches the factor ψ' is equal to 0.5, then the critical buckling length becomes:

$$N_{cr\,asym} = \frac{\pi^2 EI}{s^2} \qquad [3.13]$$

According to Moon the in-plane a-symmetric buckling load will be critical if the rise f meets the following condition:

$$f > 1.85\,\pi^2\, i/4 \quad \rightarrow \quad f/a > 4.565\, i/a \qquad [3.14]$$

Where i is the radius of gyration of the section: $i = (I/A)^{1/2}$

For a rectangular section with depth t the radius of the section is equal to: i = 0.289 t. Then snap through buckling will be critical if the ratio t/a is smaller than: t/a = 0.758 f/a. Generally the Fusée Céramique vaults were designed with a ratio f/a = ¼, then the asymmetrical buckling load is critical for: t/a < 0.19.

For a Fusée Céramique vault with a thickness of 110 mm and a span of *l* = 14.4 m, the ratio t/a is equal to t/a = 0.015 < 0.19. Thus for this vault the asymmetrical buckling mode is more critical than snapping through.

Comparing the results as described by Pi and Moon for respectively a circular and parabolic vault shows the effect of the curvature. For a vault, with a ratio f/a = ¼, the switch, asymmetrical buckling mode more critical than snapping through, is equal to:

Circular vault, f/a = ¼: t/a < 0.09
Parabolic vault, f/a = ¼: t/a < 0.19

Due to the curvature the switch is for a circular vault much smaller than for a parabolic vault.

§ 3.3 Snapping through, including the movement of the supports

The expression defining the critical buckling load in case the vault deforms symmetrically, suppose unmovable supports. Actually due to the thrust ties, connecting the supports, will lengthen. Due to the displacement of the supports the structure can collapse.

Three hinged truss

For a truss composed of two chords and a tie the effect of the lengthening of the tie is researched. A load F, acting at the top the beams, will shorten the beams and lengthen the tie. The structure is snapping through if the deformation of the top including the second order is equal to f. Consequently the length of the shortened chords is equal to the length of the lengthened tie. The section and Young's modulus of the chords and tie are respectively A_b, E_b, A_T and E_T.

Due to the concentrated load F the tie and beams are subjected to the following normal forces:

tie: $H = ½ F a/f$; with: $\tan\beta = f/a$ [3.15]

chord: $N = ½ F s/f = \dfrac{½ F a}{f \cos\beta}$ [3.16]

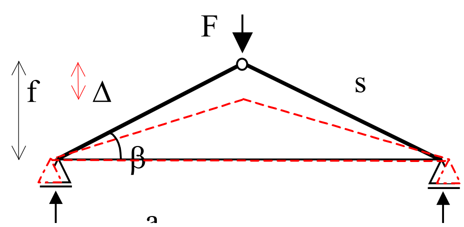

FIGURE 3.2 Truss subjected to concentrated load acting halfway the span.

According to the Theory of Elasticity the deformation of the tie, over a length a, and the chords is equal respectively to:

tie: $\Delta_t = \dfrac{F a^2}{2 f EA_T}$ [3.17]

chord: $\Delta_b = \dfrac{F s^2}{2 f EA_b} = \dfrac{F a^2}{2 f EA_b \cos^2\beta}$ [3.18]

Assuming the vertical deformation of the centre at the top is equal to Δ, the sagging is calculated using the equilibrium of the external and internal work:

$$½ F \Delta = 2 (½ H \Delta_T + ½ N \Delta_b)$$

Substitute H and N and divide the expression by F:

$$\Delta = \frac{F a^3}{2 f^2} \left(\frac{1}{EA_T} + \frac{(s/a)^3}{EA_b} \right) \rightarrow \Delta = \frac{F a^3 C}{2 f^2 EA_b} \quad [3.19]$$

With: $C = (EA_b/EA_T + 1/\cos^3\beta)$

For a small ratio f/a the length of the chord is slightly larger than the length of the tie then $s/a \approx 1$ and $\cos\beta \approx 1$. In practice the stiffness of the tie is often much than the stiffness of the chord, thus $EA_T <$ EA_b. Consequently the factor C will be larger than 1.

For $s/a \approx 1$ the factor C is: $EA_T/EA_b = 1$: $C = 2$

$EA_T/EA_b = ½$: $C = 3$

$EA_T/EA_b = ¼$: $C = 5$

The deformation of the ties increases the vertical displacement of the top and can cause snapping through. To be safe the effect of horizontal deformations of the supports must be included for swallow arches and vaults. For domes with a single grid the ratio f/a can be very small, but for arches and cylindrical vaults the ratio f/a is generally not smaller than ¼ . Then the asymmetrical buckling mode will be more critical.

For a truss composed of a tie and two chords the effect of the ratio t/a will be studied. Due to the deformation of the top with Δ the rise decreases with $f' = f - \Delta$, thus the normal forces increase:

$$\text{Tie:}\quad dH = \frac{F\,a}{2\,f}\left(\frac{1}{(1-\Delta/f)} - 1\right) \rightarrow dH = \frac{F\,a\,\Delta/f}{2\,f\,(1-\Delta/f)} \qquad [3.20]$$

$$\text{Chords:}\quad dN = \frac{F\,s}{2\,f}\left(\frac{1}{(1-\Delta/f)} - 1\right) \rightarrow dN = \frac{F\,s\,\Delta/f}{2\,f\,(1-\Delta/f)} \qquad [3.21]$$

Due to the increase of the forces with dH and dN the top will deform further with D/m. Substituting dH, dN and the rise $f' = f\,(1-\Delta/f)$ into [3.19] gives:

$$\Delta/m = \frac{F\,a^3\,C\,D/f}{2\,f^2\,EA_b\,(1-\Delta/f)^3} \qquad \text{with: } C = (E_bA_b/E_tA_T + 1/\cos^3\beta) \qquad [3.22]$$

Substituting Δ [3.19] and Δ/m [3.22] into $m = \Delta/(\Delta/m)$ gives: $m = \dfrac{(1-\Delta/f)^3}{\Delta/f}$

The structure fails in case Δ/m is larger than Δ, the maximum deformation follows from the condition $m > 1$:

$$m = \frac{(1-\Delta/f)^3}{\Delta/f} > 1 \rightarrow \Delta/f < 0.32$$

The maximum load F_u follows from $\Delta < 0.32 \times f$, substituting $\Delta < 0.32 \times f$ into [3.19] gives:

$$\frac{F_u\,a^3\,C}{2\,f^2\,EA_b} < 0.32\,f$$

$$F_u = \frac{0.64\,f^3\,EA_b}{a^3\,C} \qquad \text{with: } C = (EA_b/EA_T + 1/\cos^3\beta) \qquad [3.23]$$

Due to the critical load the normal force acting at the chord is equal $N_u = ½\,F_u/\sin\beta$. Substituting F_u according to [3.23] and $\tan\beta = f/a$ into this expression gives:

$$N_u = \frac{0.32\,f^2}{a^2\cos\beta} \times \frac{EA_b}{C} \qquad \text{with: } C = (EA_b/EA_T + 1/\cos^3\beta) \qquad [3.24]$$

For a truss the force acting at the chords is increasing linear with 1/f. If the angle β is very small, then the normal force acting at the chords will very large. Probably one of the chords fails by buckling. For the chords the buckling force N_{cr} follows from [3.2]. Substituting $s = a/\cos\beta$ into [3.2] gives the asymmetrical buckling force:

$$N_{cr} = \pi^2\,EI_b\cos^2\beta/a^2 \qquad [3.25]$$

The structure will fail by asymmetrical buckling load if the critical buckling force according to [3.24] is larger than the critical buckling force according to [3.25].

$$\frac{0.32\, f^2\, EA_b}{a^2\, C \cos\beta} > \frac{\pi^2\, EA_b\,(I/A_b)\cos^2\beta}{a^2} \quad\rightarrow\quad (f/a)^2 > \frac{\pi^2 C\,(I/A_b)\cos^3\beta}{0.32\, a}$$

For a rectangular section the ratio I/A_b is equal to $I/A_b = t^2/12$. Substituting I/A_b gives:

$$f/a > \frac{t/a \times [\cos^3\beta\, EA_b/EA_T + 1]^{1/2}}{0.624} \quad\rightarrow\quad t/a < \frac{0.624\, f/a}{(\cos^3\beta\, EA_b/EA_T + 1)^{1/2}}$$

If f/a is small then s will slightly larger than a, so $s/a \approx 1$ and $\cos\beta \approx 1$.
In practice the ratio EA_b/EA_T will be larger than 1. Decreasing the stiffness of the tie will increase the ratio EA_b/EA_t. Then the asymmetrical buckling mode will be critical for a smaller value of the ratio t/a. Generally the Fusée Céramique vaults were designed with a rise equal to f = ¼ a. Substituting f/a = ¼ gives the following values for the switch, the asymmetrical buckling will be critical if:

$EA_b/EA_T = 1$:	$t/a < 0.44\, f/a$	for f = ¼ a:	$t/a < 0.110$
$EA_b/EA_T = 2$:	$t/a < 0.35\, f/a$	for f = ¼ a:	$t/a < 0.088$
$EA_b/EA_T = 4$:	$t/a < 0.28\, f/a$	for f = ¼ a:	$t/a < 0.070$

The deformation of supports will decrease the symmetrical buckling force substantially. In practice the swallow Fusée Céramique vaults were designed with a ratio f/a = ¼ and a thickness of 110 mm for a span up to 15 m. For a vault with a span 2 a = 15 m the ratio t/a is equal to: t/a > 110/7500 = 0.015. This ratio is much smaller than the switch; the asymmetrical buckling mode is more critical than the snapping-through failure mode.

§ 3.4 Two hinged vaults.

Two hinged vaults are stiffer than three hinged vaults. For a trough vault, subjected to a concentrated load at the top, the effect of fixing the joint at the top is studied. The structure is composed of the vault with a section A_b and Young's modulus E_b and a tie with a section A_t and Young's modulus E_t. The trough vault with a fixed joint at the top is statically indeterminate. The thrust H follows from the displacement of the roller support defined separately for the tie and the vault subjected to the concentrated load and thrust.

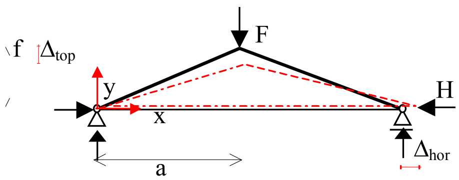

FIGURE 3.3 Trough vault subjected to a concentrated load acting at the top.

The lengthening of the tie due to the thrust H is equal to: $\Delta_T = \dfrac{2\, H\, a}{EA_T}$

Generally for the deformation of vaults the effect of the shear forces is quite small, then the deformation of the structure at a chosen point can be defined with the following expression:

$$\Delta = \int \frac{M_x M'_x\, ds}{EI} + \int \frac{N_x N'_x\, ds}{EA} \qquad [3.26]$$

With: M_x is the bending moment and N_x is the normal force acting in an element due to the load; M'_x is the bending moment and N'_x is the normal force acting in an element due to a concentrated force F = 1 acting at the chosen point, parallel to the deformation;

Due to the concentrated load F acting at the top the vault is subjected to:
a bending moment, $M_x = ½ F x$, a shear force $V_x = ½ F$ and a normal force $N_x = ½ F \sin \beta$.
Due to the thrust H acting at the supports the vault is subjected to:
a bending moment $M_x = H x f/a$, a shear force $V_x = H \sin \beta$ and a normal force $N_x = H/\cos \beta$.
Successively the deformation of the facetted beam is defined at the top and at the roller support for the load F and the thrust.

Horizontal deformation of the roller support of the facetted beam due to the concentrated load F

Due to the concentrated load F the roller support of the beam will move outward. Substituting $M_x = ½ F x$, $M'_x = x f/a$, $N_x = ½ F \sin \beta$, $N'_x = 1/\cos \beta$ and $ds = dx/\cos \beta$ into [3.26] gives:

$$\Delta_b F_{hor} = \frac{2 F \int_0^a ½ f/a\, x^2 dx}{EI_b \cos \beta} - \frac{2 F \int_0^a ½ \sin \beta\, dx}{EA_b \cos^2 \beta}$$

Integrating between x = 0 and x = a and substituting $\tan \beta = f/a$ gives:

$$\Delta_b F_{hor} = \frac{F f a^2}{3 EI_b \cos \beta} - \frac{F f}{EA_b \cos \beta} \qquad [3.27]$$

Horizontal deformation of the roller support of the facetted beam due to the thrust

Due to the thrust H the roller support of the beam will move horizontally. Substituting $M_x = H x f/a$, $M'_x = x f/a$, $N_x = H/\cos b$, $N'_x = 1/\cos \beta$ and $ds = dx/\cos \beta$ into (3.26) gives:

$$\Delta_b H_{hor} = \frac{2 H \int_0^a f^2/a^2\, x^2 dx}{EI_b \cos \beta} + \frac{2 H \int_0^a dx}{EA_b \cos^3 \beta}$$

Integrating between x = 0 and x = a gives:

$$\Delta_b H_{hor} = \frac{2 H a f^2}{3 EI_b \cos \beta} + \frac{2 H a}{EA_b \cos^3 \beta} \qquad [3.28]$$

Thrust

The thrust H follows from the deformation of the roller support: $\Delta_{b F hor} - \Delta_{b H hor} = \Delta_{tie}$

$$\frac{2 H a f^2}{3 EI_b \cos \beta} + \frac{2 H a}{EA_b \cos^3 \beta} + \frac{2 H a}{EA_t} = \frac{F f a^2}{3 EI_b \cos \beta} - \frac{F f}{EA_b \cos \beta}$$

$$H = \tfrac{1}{2} F (a/f)\, C \quad \text{with: } C = \frac{1 - 3 \times I/(A a^2)}{1 + 3 I_b/(A_b f^2) \times (1/\cos^2\beta + \cos\beta\, EA_b/EA_t)} \qquad [3.29]$$

In practice the stiffness of the beam is larger than the stiffness of the tie: $EA_b > EA_t$, then the factor C will be smaller than 1: $C < 1$. For a rectangular section the ratio I_b/A_b is equal to $I_b/A_b = t^2/12$. Then the factor C is equal to:

$$C = \frac{1 - \tfrac{1}{4}(t/a)^2}{1 + \tfrac{1}{4}(t/a)^2 (a/f)^2 (1/\cos^2\beta + \cos\beta\, EA_b/EA_t)} \qquad [3.29']$$

Next the vertical deformation of the structure is calculated for the facetted beam subjected to a vertical load acting at the top and the thrust.

Vertical deformation of the vault due to the concentrated load acting at the top.

Substituting $M_x = \tfrac{1}{2} Fx$, $M'_x = \tfrac{1}{2} x$, $N_x = \tfrac{1}{2} F \sin b$, $N'_x = \tfrac{1}{2} \sin\beta$ and $ds = dx/\cos\beta$ into [3.26] gives:

$$\Delta_{b\,F\,vert} = \frac{2\,F \int^a \tfrac{1}{4} x^2 dx}{EI_b \cos\beta} + \frac{2\,F \int^a \tfrac{1}{4} \sin^2\beta\, dx}{EA_b \cos\beta}$$

Integrating between x = 0 and x = a gives:

$$\Delta_b F_{vert} = \frac{F a^3}{6\, EI_b \cos\beta} + \frac{F f^2 \cos\beta}{2\, EA_b\, a} \qquad [3.30]$$

Vertical deformation of the vault due to the thrust

Due to the thrust H the top will deform vertically. Substituting $M_x = H \times f/a$, $M'_x = \tfrac{1}{2} x$, $N_x = H/\cos\beta$, $N'_x = \tfrac{1}{2} \sin\beta$ and $ds = dx/\cos\beta$ into [3.26] gives:

$$\Delta_{b\,H\,vert} = \frac{2\,H \int^a \tfrac{1}{2} f/a . x^2 dx}{EI_b \cos\beta} - \frac{2\,H \int^a \tfrac{1}{2} \sin\beta/\cos\beta\, dx}{EA_b \cos\beta}$$

Integrating between x = 0 and x = a gives:

$$\Delta_{b\,H\,vert} = \frac{H f a^2}{3\, EI_b \cos\beta} - \frac{H f}{EA_b \cos\beta} \qquad [3.31]$$

The deformat ion of the vault at the top

The deformation of the structure at the top follows from: $\Delta_{top} = \Delta_{b\,F\,vert} - \Delta_{b\,H\,vert}$
Substituting [3.30] and [3.31] gives:

$$\Delta_{top} = \frac{F a^3}{6\, EI_b \cos\beta} + \frac{F f^2 \cos\beta}{2\, EA_b\, a} - \frac{H f a^2 [1 - 3 I_b/(A_b a^2)]}{3\, EI_b \cos\beta}$$

Substituting H gives:

$$\Delta_{top} = \frac{F a^3}{6\, EI_b \cos\beta} \left\{ 1 - C + \frac{3 I_b f^2 \cos^2\beta}{A_b a^4} + \frac{3 I_b C}{A_b a^2} \right\} \qquad [3.32]$$

For a rectangular section the ratio I_b/A_b is equal to $I_b/A_b = t^2/12$. Then the deformation at the top is equal to:

$$\Delta_{top} = \frac{2 F a (a/t)^2 \{ 1 - C + ¼ (t/a)^2 (f/a)^2 \cos^2\beta + ¼ C \times (t/a)^2 \}}{EA_b \cos\beta} \quad \rightarrow$$

$$\frac{\Delta_{top}}{a} = \frac{2 F \{ (1 - C) (a/t)^2 + ¼ (f/a)^2 \cos^2\beta + ¼ C \}}{EA_b \cos\beta} \qquad [3.32']$$

The factor C shows the effect of the thrust. For C = 0 the structure is a section-active structure. The structure will fail if the bending stress exceeds the ultimate stress. Substitute C = 0 into expression (3.32') to define the deformation of the section-active structure:

$$\frac{\Delta_{top}}{a} = \frac{2 F (a/t)^2}{EA_b \cos\beta} + \frac{F (f/a)^2 \cos\beta}{2 EA_b}$$

For C = 1 the structure is a form-active structure. Substitute C = 1 into expression [3.32'] to define the deformation of the form-active structure:

$$\frac{\Delta_{top}}{a} = \frac{F [(f/a)^2 \cos^2\beta + 1]}{2 EA_b \cos\beta}$$

For the form-active structure with C = 1 decreasing the ratio f/a will increase the thrust. To prevent asymmetrical buckling the normal stress due to normal load has to be smaller than $1/n_{cr}$ times the buckling stress defined with [3.25] for the chord:

$$\sigma = \frac{N}{A_b} < \frac{\pi^2 EI_b \cos^2\beta}{n_{cr} A_b a^2} \quad \rightarrow \quad \frac{I_b}{A_b} > \frac{n_{cr} \sigma a^2}{E_b \pi^2 \cos^2\beta}$$

For a rectangular section the ratio I_b/A_b is equal to $I_b/A_b = {}^1/_{12}\, t^2$, then the maximum stress follows from:

$$\sigma_c < \frac{\pi^2 E_b (t/a)^2 \cos^2\beta}{12 n_{cr}}$$

Halfway the twentieth century structures of concrete were designed with a ultimate stress equal to σ_c = 8.0 MPa, a safety factor n_{cr} = 5 and a stiffness equal to $E_c = 21\times10^3$ MPa. For these structures the ratio t/a had to be: $t/a > {}^1/_{21}$. For vaults with a smaller ratio t/a asymmetrical buckling will be critical.

To prevent asymmetrical buckling of the structure the minimal stiffness follows for the chord from [3.25]:

$$N_{cr} = \frac{\pi^2 EI_b \cos^2\beta}{a^2} > \frac{n_{cr} H}{\cos\beta} \quad \rightarrow \quad EI_b > \frac{½\, n_{cr} F (a/f) a^2}{\pi^2 \cos^3\beta}$$

For a rectangular section the ratio I_b/A_b is equal to $I_b/A_b = {}^1/_{12}\, t^2$ the minimal stiffness follows from:

$$EA_b > \frac{6 n_{cr} F (a/f) \times (a/t)^2}{\pi^2 \cos^3\beta}$$

Substituting this stiffness into expression [3.32] gives for a form active structure with C = 1 the following deformation at the top:

$$\frac{\Delta_{top}}{a} < \frac{\pi^2 \cos^2\beta\, (t/a)^2 \times (f/a)\, [(f/a)^2 \cos^2\beta + 1]}{12 n}$$

For f/a = ¼ and t/a = 1/65 the deformation of the vault is equal to Δ_{top}/a = 0.00005 and Δ_{top}/f = 0.0002. As showed before trusses do not fail by snapping through if the deformation is smaller then Δ/f < 0.32; this vault will not fail by snapping through.

Non-linear analysis for validation

The following analysis shows the effect of the parameters (rise, span, thickness, ultimate stress and stiffness) for the trough vault subjected to a concentrated force F acting at the top.Due to the normal force the vault is subjected to a normal stress equal to: $\sigma = H/\cos\beta$. Substituting $H = ½ F C (a/f)$ gives:

$$\sigma = \frac{½ F (a/f) C}{A_b \cos\beta} \quad \text{with:} \quad C = \frac{1 - ¼ (t/a)^2}{1 + ¼ (t/a)^2 (a/f)^2 (1/\cos^2\beta + \cos\beta \, EA_b/EA_t)} \qquad [3.29']$$

To prevent asymmetrical buckling the compressive stress must be smaller than the buckling stress:

$$\sigma_c = \frac{½ F (a/f) C}{A_b \cos\beta} < \frac{\pi^2 E_b (t/a)^2 \cos^2\beta}{12\, n_{cr}} \quad \text{with } n_{cr} > 1 \qquad [3.33]$$

Due to the bending moment the structure is subjected to bending stresses:

$$\sigma = \frac{½ F a (1 - C)}{(A_b t/6)} \times \frac{n_{cr}}{(n_{cr} - 1)} \quad \text{with } n_{cr} > 1$$

The stress due to the bending and compression must be smaller than the ultimate stress σ_u:

$$\sigma = \frac{½ F (1 - C) \times 6 (a/t)}{A_b} \times \frac{n_{cr}}{(n_{cr} - 1)} + \frac{½ F C (a/f)}{A_b \cos\beta} < \sigma_u \qquad [3.34]$$

Next the deformation is defined with [3.32].

For a vault, with parameters t/a, f/a, s_u, E_b, E_t and EA_b/EA_t, the deformation can be defined with the following procedure:

- Define the factor C with [3.29];
- Define the ultimate buckling stress [3.33];
- Define and check the maximum normal stress with [3.34];
- Define the deformation with [3.32].

THe maximum load F follows from the check for the ultimate buckling stress and the check for the normal and bending stress. To prevent asymmetrical buckling the stress due to the normal force is at maximum: To prevent asymmetrical buckling the compressive stress must be smaller than the buckling stress [3.33], then the maximum load follows from:

$$\sigma = ½ F/A_b < \frac{\pi^2 E_b (t/a)^2 \times (f/a) \cos^3\beta}{12\, n\, C} \qquad [3.33']$$

The stress due to the bending and compression must be smaller than the ultimate stress σ_u [3.34], then the maximum load follows from:

$$\sigma = ½ F/A_b < \frac{\sigma_u}{6 (1 - C) \times (a/t) ´ n/(n-1) + C (a/f)/\cos\beta} \qquad [3.34']$$

The maximum load follows from the minimum value of the stress $\sigma = ½ F/A_b$ defined with [3.33'] and [3.34']. Substituting the maximum load into [3.32'] gives the deformation at the top.

$$\frac{\Delta_{top}}{a} = \frac{(½ F/A_b) \{4 (1 - C) (a/t)^2 + (f/a)^2 \cos^2\beta + C\}}{E_b \cos\beta} \qquad [3.32']$$

Example

Halfway the twentieth century structures of concrete were designed with a ultimate stress equal to s_c = 8.0 MPa, a safety factor n = 5 and a stiffness $E_c = 21\times 10^3$ MPa. For this structure the load transfer and deformation is defined for t/a = 1/30, f/a = ¼, $EA_b/EA_t = 3$ and cos β = 0.97. Firstly the ratio C is defined [3.29']:

$$C = \frac{1 - ¼ \times (1/30)^2}{1 + ¼ \times (1/30)^2 \times 4^2 \times (1/0.97^2 + 0.97\times 3)} = 0.98$$

To prevent asymmetrical buckling the maximum load is defined with [3.33']:

$$½\, F/A_b < \frac{\pi^2 \times 21000 \times (1/30)^2 \times (1/4) \times 0.97^3}{12 \times 5 \times 0.98} = 0.89 \text{ MPa}$$

The stress due to the bending and compression must be smaller than the ultimate stress, the maximum load is defined with [3.34']:

$$½\, F/A_b < \frac{8.0}{(1 - 0.98) \times 6 \times 30 \times 5/(5-1) + 0.98\times 4/0.97} = 0.94 \text{ MPa}$$

Comparing both values gives the maximum load: $½\, F/A_b <$ 0.89 MPa. Substituting this load into (3.32'') gives the maximum deformation at the top:

$$\frac{\Delta_{top}}{a} = \frac{0.89 \times \{4\,(1 - 0.98)\,30^2 + (¼)^2 ´\, 0.97^2 + 0.98\}}{21000 \times 0.97} = 0.003$$

The deformation is very small, snapping through will not be critical.

Decreasing the ratio f/a

Next the ratio f/a is decreased, f/a = 1/10 and cos β = 0.995:

$$C = \frac{1 - ¼ \times (1/30)^2}{1 + ¼ \times (1/30)^2 \times 10^2 \times (1/0.995^2 + 0.995\times 3)} = 0.9$$

To prevent asymmetrical buckling the maximum load is defined with [3.33']:

$$½\, F/A_b < \frac{\pi^2 \times 21000 \times (1/30)^2 \times (1/10) \times 0.995^3}{12 \times 5 \times 0.9} = 0.42 \text{ MPa}$$

The stress due to the bending and compression must be smaller than the ultimate stress, the maximum load is defined with [3.34']:

$$½\, F/A_b < \frac{8}{6 \times (1 - 0.9) \times 30 \times 5/(5-1) + 0.9 \times 10/0.995} = 0.25 \text{ MPa}$$

Comparing both values gives for the maximum load: $½\, F/A_b <$ 0.25 Mpa. Substituting this load into (3.32'') gives the maximum deformation at the top:

$$\frac{\Delta_{top}}{a} = \frac{0.25 \times \{4 \times (1 - 0.9) \times 30^2 + (1/10)^2 \times 0.995^2 + 0.9\}}{21000 \times 0.995} = 0.004$$

The deformation is very small, snapping through will be not critical. Decreasing the ratio f/a will decrease the maximum load as well.
To show the effect of the parameters t/a and f/a the following graph is constructed for the trough vault subjected to a concentrated load at the top where $n_{cr} = 5$, $\sigma_u = 8$ MPa, $EA_b/EA_t = 3$ and E_b = 21000 MPa. For f/a = 0 the thrust is zero, the vault transfers the load as a plate subjected by bending.

Decreasing the ratio f/a increases the thrust and decreases the maximum load. Decreasing the depth t will decrease the asymmetrical buckling resistance and decrease the maximum load too.

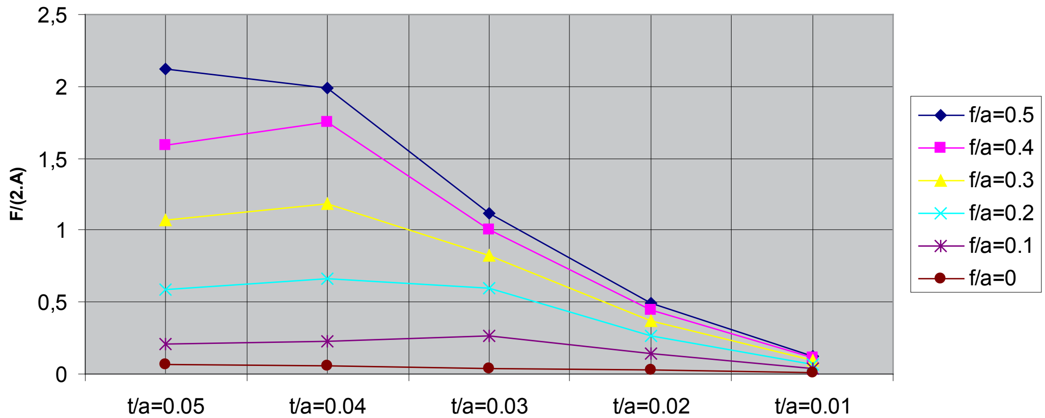

FIGURE 3.4 Graph showing the maximum load ½ F/A [MPa] for the trough vault subjected to concentrated load acting at the top for f/a varying from 0 to 0.5, with n_{cr} = 5, σ_u = 8 MPa, EA_b/EA_T = 3 and E_b = 21000 MPa.

§ 3.5 Effect of the hangers

Adding a hanger between the top and tie will increase the resistance significantly. The tie, tensioned by the thrust, will push the top upward if the top is sagging and prevent the structure snapping through. The hanger is loaded by a compressive normal force and must be stiff enough to resist the buckling.

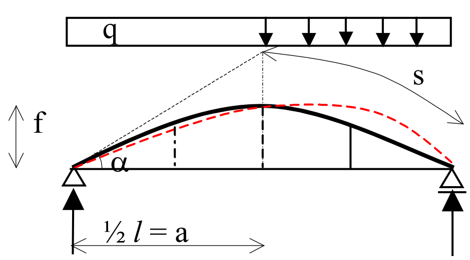

FIGURE 3.5 Two hinged arch, subjected to equally distributed, with n = 3 hangers

Palkowski [Pal12] researched the increase of critical buckling load due to the use of ties and hangers. The number of the hangers as well as the position and inclination affect the critical buckling load for a parabolic pin-ended arch subjected to an equally distributed load.

The critical buckling load for the asymmetric mode follows from:

$$q_{cr} = \frac{k\,EI}{l^3} \quad [3.35]$$

For a parabolic arch subjected to an equally distributed load the thrust is equal to:

$$H = \frac{q\,l^2}{8\,f} \qquad [3.36]$$

Substituting q_{cr} [3.35] into [3.36] results in:

$$H_{cr} = \frac{k\,EI}{8\,f\,l} \qquad [3.37]$$

The maximum normal force acting at the vault at the supports is equal to:

$$N = H/\cos\alpha \qquad [3.39]$$

Where α is the angle with the horizontal axis at the support, with $\tan\alpha = y' = 2\,f/a$
Substituting H_{cr} [3.37] into [3.39] gives:

$$N_{cr} = \frac{k\,EI}{8\,f\,l\,\cos\alpha} \qquad [3.40]$$

The critical buckling force according to Euler [3.3] is equal to: $N_{cr} = \pi^2\,EI/\,(\psi\,s)^2$ The factor ψ follows from expression [3.40] and [3.3]:

$$\frac{k\,EI}{8\,f\,l\,\cos\alpha} = \frac{\pi^2\,EI}{(\psi\,s)^2} \quad \rightarrow \quad \psi = \frac{\pi\,l}{s} \times \frac{(8\,f\cos\alpha)^{1/2}}{(k\,l)^{1/2}} \qquad [3.41]$$

Table 3.5 shows an increase of the buckling load q_{cr} and a decrease of the factor ψ, if the number of hangers is increased from m = 1 to m = 3. Consequently increasing the number of hangers will increase the capacity to resist loads.

	f/l = 0.1	f/l = 0.2	f/l = 0.3	f/l = 0.4	f/l = 0.5
ratio s/l	0.513	0.549	0.602	0.667	0.739
cos α	0.928	0.781	0.64	0.53	0.447
m = 1 factor k	28.7	45.5	47.2	44.3	38.6
factor ψ	0.99	0.95	0.94	0.92	0.91
m = 3 factor k	101.6	112.4	90.9	68.2	51.5
factor ψ	0.52	0.6	0.68	0.74	0.79

TABLE 3.5 Factor k and ψ for a pin-ended arch with one vertical hanger (m = 1) and three vertical hangers (m = 3) for varying ratios of the rise to span f/l [Pal12]

§ 3.6 The critical buckling load for arches with a convex tie.

As mentioned before Palkowski [Pal12] researched the increase of critical buckling load due to the use of ties and hangers. Due to the asymmetrical deformation of the arch the hangers at the side deforming upward are tensioned but the hangers at the side deforming downward are compressed. Generally the slender hangers can not resist compressive forces. To resist a compressive force the hangers must pre-tensioned, for example by curving the tie upward. The increase of the critical buckling force for a parabolic pin-ended arch with a tie curved upward is studied. The arch is subjected to an equally distributed load. The tie is curved upward with three hangers. The curvature of the tie is equal to c f at the centre. Due to the curvature of the tie the hangers are subjected to the forces S. For a parabolic arch subjected to an equally distributed load and the force S the thrust follows from:

$$H = \frac{q\,a^2}{2\,f\,(1\text{-}c)} + \frac{S\,a}{f\,(1\text{-}c)} \qquad [3.42]$$

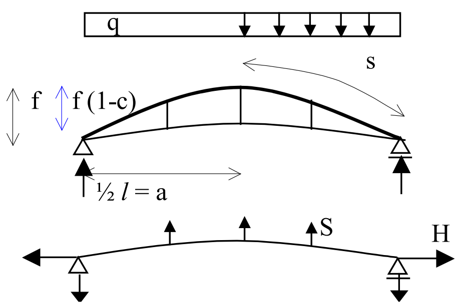

FIGURE 3.6 Two hinged arch, with 3 hangers and a convex tie, subjected to an equally distributed load.

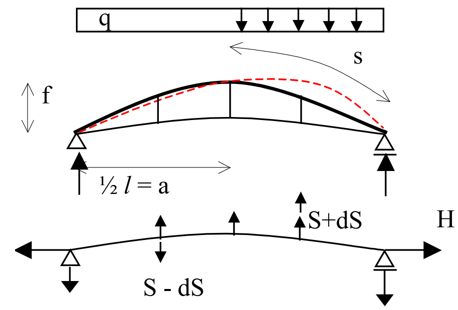

FIGURE 3.7 Two hinged arch, with 3 hangers and a convex tie, subjected to an equally distributed load, deforming asymmetrically.

The tie can not stand any bending moment. The force S follows from the equilibrium of the bending moment at the top for x = 0, $M_{x=0} = 0$.

$$M_{x=0} = {}^3/_2\,S\,a - ½\,a\,S - H\,c\,f = 0 \quad \rightarrow \quad S = H\,c\,f/a \qquad [3.43]$$

Next the force S [3.43] is substituted into expression [3.42]:

$$H = \frac{q_{cr}\,a^2}{2\,f\,(1\text{-}c)} + \frac{H\,c\,f\,a}{a\,f\,(1\text{-}c)} \quad \rightarrow \quad H = \frac{q_{cr}\,a^2}{2\,f\,(1\text{-}\,2\,c)} \qquad [3.44]$$

Due to the buckling load the arch is assumed to deform asymmetrical. At a quarter of the span the deformation is equal to du. Further the deformation due to the buckling load is assumed to be sinusoidal with:$du_x = du \sin(\pi\,x/s)$ Due to the deformation the forces acting on the outward hangers increase with dS = k du, k is resiliency of the tie. The resiliency follows from the equilibrium of the bending moments acting on the tie:

$$M = H\,du - ½\,dS\,½\,a = 0 \quad \rightarrow \quad dS = 4\,H\,du/a$$

Substituting dS = k du gives: $k = 4\,H/a$ [3.45]

Due to the deformation $u = du \sin(\pi\,x/s)$ the arch is subjected to bending moments.

Due to the anti-metrical loads dS acting at the hangers the arch is subjected to reaction forces acting at the supports ½ dS. the thrust dH follows from the bending moment at the top:

$$dH\,f = dS\ ½\,a - ½\,dS\,a \quad \rightarrow \quad dH = 0$$

At a quarter of the span the bending moment due to the load dS is equal to M = ¼ a dS. Due to the deformation and the force dS the bending moment acting on the arch is at a quarter of the span:

$$M_{x=a/2} = N\,du - ¼\,a\,dS$$

With dS = 4 H du/a the bending moment acting at the arch is: $M_{x=a/2} = N\,du - H\,du$

Due to this moment the deformation of the arch increases with du/n_{cr}:

$$\frac{du}{n_{cr}} = \frac{s^2\,(N - H)\,du}{\pi^2\,EI} \qquad [3.46]$$

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For the critical buckling load q_{cr} the normal force increases to N_{cr} , and the deformation will increases to n_{cr} (du/n_{cr}):

$$\frac{n_{cr}\,du}{n_{cr}} = \frac{s^2 (N_{cr} - H_{cr})\,du}{\pi^2 EI} \quad \rightarrow \quad N_{cr} = \frac{\pi^2 EI}{s^2} + H_{cr} \qquad [3.47]$$

At the support the critical normal force is: $H_{cr} = N_{cr} \cos \alpha$. Substituting the thrust into [3.47] gives for the buckling load:

$$N_{cr} = \frac{\pi^2 EI}{(1-\cos \alpha)\, s^2} \qquad [3.48]$$

The critical buckling force according to Euler [3.3] is equal to: $N_{cr} = \pi^2 EI/ (\psi\, s)^2$ The factor ψ follows from expression [3.3] and [3.48]:

$$\frac{\pi^2 EI}{(\psi\, s)^2} = \frac{\pi^2 EI}{(1-\cos \alpha)\, s^2} \quad \rightarrow \quad \psi = [1-\cos \alpha]^{1/2} \qquad [3.49]$$

For $f/l = 0.1$ the angle α follows from $\tan \alpha = 0.4$; substituting $\cos \alpha = 0.928$ into [3.49] gives the reduction factor ψ is equal to: $\psi = (1\text{-}0.928)^{1/2} = 0.14$. So for $f/l = 0.1$ the factor ψ is smaller than 0.5. The buckling mode will increase so the buckling force of the arch with a length ½ s will be decisive, $\psi = 0.5$, this value is approximately equal to the value found in table 3.5.

§ 3.7 The critical buckling load for arches connected with three hangers tensioned at one side.

Due to the asymmetrical deformation of the arch the hangers at the side deforming upward are tensioned but the hangers at the side deforming downward are compressed. Generally the slender hangers can not resist compressive forces. Neglecting the compressed hangers the buckling of the arch or vault is only restricted by the hangers at one side. The increase of the critical buckling force for a parabolic pin-ended arch due to tensioned hanger at the upward curved side is studied.

The arch is subjected to an equally distributed load. Three hangers connect the tie with the arch. For a parabolic arch subjected to an equally distributed load the thrust follows from

$$M_{x=a} = 0: \quad H = {}^{1}/_{8}\, q\, l^2/f$$

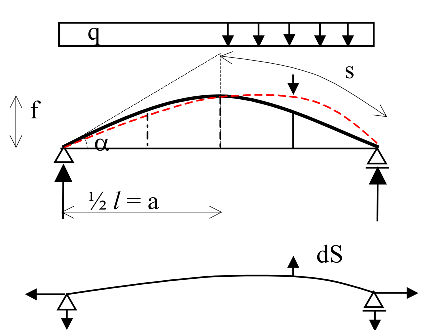

FIGURE 3.8 wo hinged arch, subjected to equally distributed, with n = 3 hangers and a convex tie

To define the buckling load the arch is assumed to deforms asymmetrical. For x = ½ a at a quarter

of the span the deformation is equal to u. Further the deformation of the arch is assumed to be sinusoidal with: $u_x = u \sin(\pi x/s)$.
Due to the curvature of the arch the hanger, between the tie and the side of the arch deforming upward, is subjected to a tensile force dS. The hanger, between the tie and the arch deforming downward, is subjected to a compressive force, this slender tie will buckle and not transfer a normal force. The force acting on the tensioned hanger depends on the resiliency k of the tie: $dS = k\,du$.
The tie can not stand bending moments. The force dS follows from the equilibrium of the bending moment at the top for $x = ½ a$, $M_{x=a/2} = 0$.

$$M_{x=a/2} = ¾\, dS\,(½ a) - H u = 0 \quad \rightarrow \quad dS = {}^8/_3\, H u/a \qquad [3.50]$$

The resiliency k of the tie follows from S = k u, thus: $k = S/u = {}^8/_3\, H/a$ [3.51]

Due to the force dS the arch is subjected to bending, the thrust follows from the equilibrium of bending moments at the top:

$$dH\, f = ¾\, a\, dS - ½\, a\, dS \quad \rightarrow \quad dH = ¼\, dS\; a/f \qquad [3.52]$$

At a quarter of the span, for x = ½ a from the top, the bending moment is:

$$M_{x=½a} = ¼\, dS\,(½ a) - (¼\, dS\, a/f)\, ¼ f \quad \leftarrow \quad M_{x=½a} = {}^1/_{16}\, a\, dS$$

With $dS = k\,u = ({}^8/_3\, H/a)\,u$ the bending moment is equal to: $M_{x=½a} = {}^1/_6\, H u$

Due to the deformation u = du sin(πx/s) and the force dS the arch is subjected to bending moments.
At a quarter of the span the bending moment is equal to:

$$M_{x=½a} = N u - {}^1/_{16}\, dS\, a$$

With $dS = k\,u = ({}^8/_3\, H/a)\,u$ the bending moment due to the deformation and force dS is equal to:

$$M_{x=½a} = N u - {}^1/_6\, H u$$

Due to the bending moments the deformation of the arch increases with u/n_{cr} with:

$$\frac{du}{n_{cr}} = \frac{s^2 (N\,du - {}^1/_6\, H u)}{\pi^2 EI} \qquad [3.53]$$

For the critical buckling load q_{cr} the normal force will increase to N_{cr} and the deformation will increase to n_{cr} $(du/n_{cr}) = du$:

$$du = \frac{s^2 (N_{cr}\,du - {}^1/_6\, H_{cr}\,du)}{\pi^2 EI}$$

The buckling load is equal to: $N_{cr} = \dfrac{\pi^2 EI}{s^2} + {}^1/_6\, H_{cr}$ [3.54]

At the support the critical normal force is: $N_{cr} = H_{cr}/\cos\alpha$. Then the buckling load is equal to:

$$N_{cr} = \frac{\pi^2 EI}{(1 - {}^1/_6 \cos\alpha)\, s^2} \qquad [3.55]$$

The critical buckling force according to Euler [3.3] is equal to: $N_{cr} = \pi^2 EI/(\psi\, s)^2$ The factor ψ follows from expression [3.3] and [3.55]:

$$\frac{\pi^2 EI}{(\psi\, s)^2} = \frac{\pi^2 EI}{(1 - {}^1/_6 \cos\alpha)\, s^2} \quad \rightarrow \quad \psi = [1 - {}^1/_6 \cos\beta']^{1/2} \qquad [3.56]$$

For $f/l = {}^{1}/_{8}$ and $\cos \alpha = 0.894$ the reduction factor ψ is equal to : $\psi = (1 - 0.894/6)^{1/2} = 0.92$

The reduction of the buckling length, if only the hanger is tensioned between the tie and vault deformed upward, is much smaller than the reduction of the buckling length if the tie is curved upward to tension all hangers continuously.

§ 3.8 Conclusions

Usually the effect of the stiffness of supports on the critical buckling load is neglected. A tie joining the supports of an arch will lengthen and thus decrease the critical buckling load. Possibly neglecting the stiffness of the supports can overestimate the critical buckling load and thus cause failure by snap through. For Fusée Céramique barrel vaults the asymmetrical buckling mode will be more critical than the symmetrical buckling mode. Constructing hangers between the tie and the vault reduces the buckling length and increases the buckling force. The buckling force is increased further if the tie is convex, so the hangers between the vault and tie are tensioned continuously. For slender arches and vaults it can be efficient to curve the ties upward and tension the hangers continuously to reduce the buckling length of the arch or vault..

4 Including time dependent effects

Structures of concrete are subjected to time dependent material behaviour such as shrinkage and creep. Due to shrinkage and creep of concrete the deformations of the structure are increased. For structures composed of multiple materials, the time dependent deformations will cause stresses into these materials if the materials are tightly connected and this will change the load transfer. Scherpbier describes for reinforced concrete the effect of shrinkage and creep [Sch65]. This chapter describes the effect of the time dependent deformations concerning the load transfer for Fusée Céramique vaults subjected to a normal force.

§ 4.1 Time dependent effects

Shrinkage

Due to shrinkage the concrete will deform. The specific deformation of the concrete due to the shrinking is named ε_{rc}. For concrete we can distinguish between the shrinkage during the setting ε_{ca} and the shrinkage due to drying ε_{cd}. The total shrinkage is the sum of these two: $\varepsilon_{rc} = \varepsilon_{ca} + \varepsilon_{rcd}$.

Setting shrinkage

The setting shrinkage develops during the setting of the concrete just after the pouring and can be defined according to NEN-EN 1992-1-1, Euro code 2, table 3.2 [C6] with:

$$\varepsilon_{ca} = \beta_{ar} \times 2.5 \times (f_{ck}-10) \times 10^{-6}$$

With: $\beta_{ar} = 1 - e^{-0.2\sqrt{t}}$
t = the time in days
f_{ck} = characteristic cylindrical strength of concrete

For example: for C20/25 the characteristic cylindrical strength is f_{ck} = 20 MPa. The setting shrinkage is thus:

$$\varepsilon_{ca} = (1 - \varepsilon^{-0.2\sqrt{t}}) \times 2.5 \times (20-10) \times 10^{-6} = 25 \times 10^{-6} \times (1 - \varepsilon^{-0.2\sqrt{t}})$$

Shrinkage due to drying

The specific deformation of the concrete due to the shrinkage due to drying depends on the humidity of the environment and the quality of the concrete. The shrinkage is non-linear during the time and can be described with:

$$\varepsilon_{r\,cd\ t} = \beta_{dr}(t/t_o)\, k_h\, \varepsilon_{rcd\ t=\infty}$$

With: k_h depends on the fictional thickness h_0, as shown in table 2: $h_0 = 2A_c/u$ and A_c = area, u = perimeter
The ratio $\beta_{dr}(t/t_o)$ follows from:

$$\beta_{dr}(t/t_o) = \frac{(t - t_o)}{(t-t_0) + 0.04 \times h_o^{3/2}}$$

t = time in days, t_o = time of the start of the drying usually when the concrete is cured.

Table 4.1 and table 4.2 show for the shrinkage the effect of the relative humidity and thickness.

Environment	Relative humidity RH	Shrinkage C20/25
dry environment, for example an interior space	60%	$\varepsilon_{rcd} = 0.49 \times 10^{-3}$
exterior	80%	$\varepsilon_{rcd} = 0.30 \times 10^{-r}$
In a humid environment	90%	$\varepsilon_{rcd} = 0.17 \times 10^{-3}$
In water	100%	$\varepsilon_{rcd} = 0$

TABLE 4.1 The shrinkage of concrete ε_{rcd} for C20/25, t = ∞ according to table 3.2 Euro code 2 [C6]

h_0	k_0
100	1.0
200	0.85
300	0.75
> 500	0.7

TABLE 4.2 The ratio k_0 for the thickness h_0 according to table 3.3 Euro code 2 [C6]

Due to the shrinkage the concrete will deform, but the fusées and steel reinforcement will not shrink. For the Fusée Céramique vaults the concrete is bonded well to the fusées and reinforcement. So the deformations of the fusées, reinnforcement and concrete must be equal. The shrinkage of the concrete will be compensated by internal forces which will tension the concrete and compress the fusées and reinforcement. To joint the fusée elements the cilindrical top is shoven in the rear of the next element. For a cilinder any section, not perpendicular to the main axis, follows an ellipsis. Due to the curvature of the vault the cilindrical top of a fusée element is connected at a few concentrated points with the cilindrical inner surface of the neighbouring element. During the construction of the vault the concrete fills the gaps partly. Due to the shrinkage of the concrete the fusées will be subjected to a compressive load. The joints, subjected to concentrated loads will deform. To include the deformation of the joints the assumption is made that due to the shrinkage of the concrete the specific deformation of the fusées is about: $\varepsilon_{rf} = 0.1 \times 10^{-3} = 0.01\%$. In a following paragraph the effect of this deformation will be analysed.

Creep

Creep is an increase of the deformation caused by a constant load during a certain time. For example a structure of concrete subjected to a compressive normal force will deform immediately elastically when the load is added. This initial deformation is named the instantaneous deformation. If the load is not changed, and held constant the deformation increases. This increased deformation is creep. The creep is increasing during the time the load acts on the structure. Thus the creep is the difference of the total deformation minus the instantaneous deformation. Generally the instantaneous deformation is named ε_0. For a long period with t = ∞ the increase of the deformation is $\Delta\varepsilon_{t=\infty} = \varepsilon_0\, \phi$, if the stress in the concrete is less than 0.45 f_{ck}.

The factor ϕ depends on the quality of the concrete, the age of the concrete, the humidity of the environment, the thickness of the sections, the ratio area/surface $h_0 = 2 A_c/u$ and the time t_o when the load is acting on the structure. The NEN-EN 1992-1-1, Euro code 2, gives tables that define the creep factor [C6]. Generally the creep factor of a structure of concrete with a relative humidity of 80% is varying from 2 to 3. The creep factor is calculated for C20/25 cement class N and a thickness of 130 mm for several values of the time of loading t_0 in the following table.

t_0	RH = 50%	RH = 80%
$t_0 = 2$	$\phi = 4.8$	$\phi = 3.6$
$t_0 = 5$	$\phi = 3.8$	$\phi = 2.9$
$t_0 = 10$	$\phi = 3.0$	$\phi = 2.2$
$t_0 = 30$	$\phi = 1.5$	$\phi = 1.2$

TABLE 4.3 C20/25 class N, creep factor according to NEN-EN 1992-1-1 Eurocode 2[C6]: figure 3.1:

If the load is removed the deformation does not return to zero immediately but is only partly reduced. Just after the removal of the load the deformation is reduced with a deformation more or less equal to the instantaneous deformation. After the removal of the load the deformation is decreased slowly. Finally only a small deformation remains. This process is shown in figure 4.1. Firstly the structure is subjected to a load at a time $t = t_1$. The specific deformation is equal to the instantaneous specific deformation ε_0. For $t = t_2$ the specific deformation has increased due to creep with $\phi_{t2}\, \varepsilon_0$. The sum of the specific instantaneous deformation and the specific deformation due to the creep is equal to $\varepsilon_t = \varepsilon_0 (1 + \phi_{t2})$. Next the load is taken away at the time t_2. The removal of the load can be modelled by loading the structure with a counter load equal to the first load. At the time t_2 the deformation is reduced with the instantaneous specific deformation ε_0. For a time $t = t_3$ the specific deformation is increased by the creep too with $\phi_{t3}\, \varepsilon_0$. After the time t_2 the specific deformation follows from:

$$\varepsilon_t = \varepsilon_0 + \phi_{t2}\, \varepsilon_0 - \varepsilon_0 - \phi_{t3}\, \varepsilon_0$$

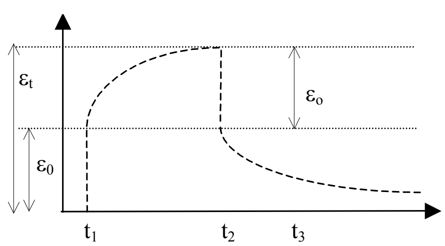

FIGURE 4.1 Instantaneous specific deformation and creep in case a structure is subjected to a load from t_1 till t_2.

For roof structures the permanent load will mostly cause the creep. In the Netherlands the live loads, caused by heavy wind, snow or rain will only act for a short period on a roof, so the increased creep due to these loads can be neglected.

§ 4.2 Structures loaded by a normal force

For structures composed of several materials the time dependent deformation due to creep and shrinkage can affect the internal distribution of the load. Firstly structures of reinforced concrete subjected to a normal load are analysed, next structures composed of fusées and concrete and structures composed of fusées and reinforced concrete are analysed.

Reinforced concrete

For a structure of reinforced concrete the effect of the time dependent deformations is generally modest. Assume a reinforced concrete structure is subjected to a permanent compressive load N. Due to this load the instantaneous specific deformation of the steel and concrete is equal to ε_0. Due to the creep of the concrete the specific deformation will rise with $\phi\, \varepsilon_0$. Then the total deformation is equal to $\varepsilon_0 (1 + \phi)$. The steel rebars and concrete are attached well, so the deformation due to creep results in a deformation of the steel too. For the sake of the compatibility an internal force F_c tensions the concrete and the rebars are compressed by an internal force F_s. The sum of the internal forces necessarily equals to zero: $\Sigma F = 0$, so $F_s = F_c$. The magnitude of the internal forces thus follows from the equations describing the equilibrium of the forces and the compatibility.

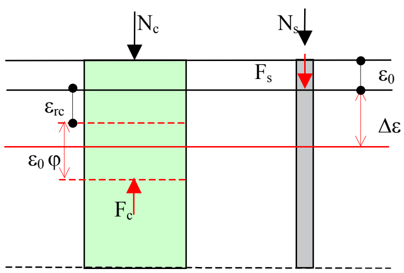

FIGURE 4.2 Deformations due to shrinkage and creep and the compensating forces F_c and F_s for reinforced concrete

The reinforced concrete structure is subjected to a normal compressive force N. At the time t = 0 the load is resisted by the concrete and rebars, the normal force in the concrete is equal to N_c and the normal force acting in the rebars is equal to N_s. The sum of these forces is equal to the load: $N_c + N_s = N$. For a symmetrically loaded structure the specific deformation of the concrete is equal to the specific deformation of the reinforcement, thus:

$$\varepsilon_c = \varepsilon_s = \varepsilon_0 \qquad [4.1]$$

According Hooke's law the forces in the concrete, fusées and reinforcement follow from:

$$N_c = AE_c\, \varepsilon_0 \quad \text{and} \quad N_s = AE_s\, \varepsilon_0$$

Substituting these expressions into [4.1]:

$$\varepsilon_0 = \frac{N_c}{AE_c} = \frac{N_s}{AE_s} \qquad [4.2]$$

Substituting these specific deformations into the expression for the equilibrium of force results in:

$$N = (AE_c + AE_s)\, \varepsilon_0$$

With this formula the deformation for t = 0 can be calculated:

$$\varepsilon_0 = \frac{N}{AE_c\,(1 + AE_s/AE_c)} \qquad [4.3]$$

Next the stress in the concrete and the reinforcement is calculated with $\sigma_c = E_c\,\varepsilon_0$ and $\sigma_s = E_s\,\varepsilon_0$.
Due to the creep and shrinkage the deformation of the structure will increase with $\Delta\varepsilon$. At time t the specific deformation of the structure is: $\varepsilon_t = \varepsilon_0 + \Delta\varepsilon$.
For the concrete the specific deformation is increased due to the creep with $\varepsilon_0\,\phi$ and due to the shrinkage with ε_{rc}. The deformation of the concrete and steel must be equal, so internal forces must equalize the differences. The concrete is loaded by an internal force F_c and the steel is subjected to an internal force F_s. Due to the internal force F_c acting on the concrete the specific deformation of the concrete decreases by: F_c/AE_c. During the time t the specific deformation resulting from force F_c is increased by creep with:

$F_c\,k\,\phi/AE_c$.

The force F_c is not constant but increasing during the time t. The factor k compensates for the time dependency of this force. Scherpbier showed that this factor is equal to k = ½ [Sch65]. The specific deformation due to the internal force F_c including the creep is equal to:

$F_c\,(1+ k\,\phi)/AE_c$.

For the concrete and the reinforcement the specific deformation is respectively equal to:

$$\varepsilon_t = \varepsilon_0 + \Delta\varepsilon = \varepsilon_0 + \varepsilon_0\,\phi + \varepsilon_{rc} - \frac{F_c\,(1+ k\,\phi)}{AE_c} \qquad [4.4]$$

$$\varepsilon_t = \varepsilon_0 + \Delta\varepsilon = \varepsilon_0 + \frac{F_s}{AE_s} \qquad [4.5]$$

The deformations are equal so the result of equation [4.4] is equal to the result of [4.5]. Furthermore the sum of the internal forces is necessarily zero, thus: $F_c = F_s = F$:

$$\varepsilon_0 + \varepsilon_0\,\phi + \varepsilon_{rc} - \frac{F\,(1+ k\,\phi)}{AE_c} = \varepsilon_0 + \frac{F}{AE_{fs}}$$

Next the force F is calculated with:

$$F = \frac{(\varepsilon_0\,\phi + \varepsilon_{rc})\,AE_c}{1 + k\,\phi + AE_c/AE_s} \qquad [4.6]$$

After time t the forces acting on the reinforced concrete and the reinforcement are respectively $N_c - F$ and $N_s + F$.

Structure composed of concrete and fusées

For a structure composed of concrete and fusées the effect of the time dependent deformation is very significant if the cross-sectional area of the fusées approaches the cross-sectional area of the concrete. Assuming that the structure is subjected to a permanent compressive load, this load will result in an instantaneous specific deformation ε_0. Creep will cause the specific deformation to increase with $\varepsilon_0\,\phi$. The total deformation is $\varepsilon_0\,(1 + \phi)$. Due to shrinkage and creep of the concrete, the structure deforms. The fusées and the concrete are attached good enough that they need to follow each other's deformation. The deformation of the concrete is greater than the deformation of the fusées. As a result the fusées are subjected to an internal compressive force F_f. The same force F_c acts on the concrete in the opposite direction, so the internal force tensions the concrete. The internal forces are necessarily equal, thus: $F_c = F_f = F$. The magnitude of the internal forces follows from the equations describing the equilibrium of the forces and the compatibility of the materials. For a reinforced concrete structure subjected to a normal compressive load the redistribution of the forces due to time dependent effects will now be described.

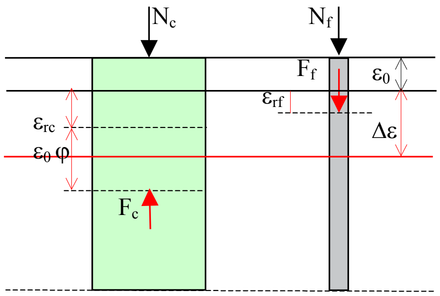

FIGURE 4.3 Deformations due to shrinkage and creep and the compensating forces for a structure of concrete and fusées.

A structure is subjected to a normal force N, due to this normal force the concrete and fusees are subjected to respectively N_c and N_F. The sum of these forces is equl to the load: $N_c + N_F = N$. For a symmetrically loaded structure the specific deformation of the concrete and fusées is equal, thus:

$$\varepsilon_c = \varepsilon_f = \varepsilon_0 \qquad [4.7]$$

According to the Theory of Elasticity the forces in the concrete and fusées are respectively:

$N_c = AE_c\,\varepsilon_c$ and $N_f = AE_f\,\varepsilon_f$

Substituting these expressions into (4.7) gives:

$$\varepsilon_0 = \frac{N_c}{AE_c} = \frac{N_f}{AE_f} \qquad [4.8]$$

Substituting these specific deformations into the expression for the equilibrium of the forces:

$$N = AE_c\,\varepsilon_0 + AE_f\,\varepsilon_0 \qquad [4.9]$$

With this expression we can calculate the immediate deformation at t = 0:

$$\varepsilon_0 = \frac{N}{AE_c\,[1 + AE_f/AE_c]} \qquad [4.10]$$

The stress in the concrete and fusées is respectively equal to: $\sigma_c = E_c\,\varepsilon_0$ and $\sigma_f = E_f\,\varepsilon_0$.

Due to creep and shrinkage the deformation of the structure will increase with $\Delta\varepsilon$. At time t the specific deformation of the structure will be equal to: $\varepsilon_t = \varepsilon_0 + \Delta\varepsilon$. For the concrete component the specific deformation increases by creep with $\varepsilon_0\,\phi$ and by shrinkage with ε_{rc}. For the fusées the specific deformation is assumed to be increased with ε_{rf} due to the deformation of the joints caused by the shrinkage of the concrete . The deformation of the concrete and fusées is equal, so the internal forces equalize the differences. An internal force F_c loads the concrete and an internal force F_f loads the fusées. The sum of these forces is necessarily zero, thus: $F_c = F_f = F$.
Due to the internal force F_c acting on the concrete the specific deformation of the concrete decreases by: F_c/AE_c.
During time t, due to this force F_c, the specific deformation increases by creep with: $F_c\,k\,\phi/AE_c$.

The force F_c is not constant but increases during the time t, the factor k compensates for the time dependency of this force. Scherpbier showed that this factor is equal to k = ½ [Sch65]. The specific deformation due to the internal force F_c including the creep is equal to:

$$\varepsilon_t = F_c\,(1+ k\,\phi)/AE_c \qquad [4.11]$$

For the concrete the specific deformation is:

$$\varepsilon_t = \varepsilon_0 + \varepsilon_0\,\phi + \varepsilon_{rc} - \frac{F_c(1+ k\,\phi)}{AE_c} \qquad [4.12]$$

For the fusées the specific deformation is:

$$\varepsilon_t = \varepsilon_0 + \varepsilon_{rf} + \frac{F_f}{AE_f} \qquad [4.13]$$

Combining [4.12] and [4.13] with $F_c = F_f = F$ gives:

$$\varepsilon_0 + \varepsilon_0\,\phi + \varepsilon_{rc} - \frac{F_c(1+ k\,\phi)}{AE_c} = \varepsilon_0 + \varepsilon_{rf} + \frac{F_r}{AE_f}$$

Next the force F is calculated with:

$$F = \frac{(\varepsilon_0\,\phi + \varepsilon_{rc} - \varepsilon_{rf})\,AE_c}{1 + k\,\phi + AE/AE_f} \qquad [4.14]$$

After time t the forces acting on the reinforced concrete and the fusees are respectively $N_c - F$ and $N_f + F$.

Example

For building Q the forces acting at a section are calculated in chapter 5. Due to the permanent load the vault is subjected to a normal force equal to N = 48.9 kN. For this example the effect of the rebars is neglected. Young's modulus and Area for a width of 1.0 m are respectively:

Concrete: E_c = 21000 MPa, A_c = 74708 mm²

Fusées: E_c = 17000 MPa, A_c = 24190 mm²

Immediately deformation, t = 0. Due to the normal force N = 48.9 kN the specific deformation is:

$$\varepsilon_0 = \frac{48900 \times 21000}{(21000 \times 74708 + 17000 \times 24190)\times 10^3} = 0.0247 \times 10^{-3}$$

The normal stresses and forces acting on respectively concrete and fusées follows from: $\sigma_x = \varepsilon_0\,E_x$

Concrete: $\sigma_c = 0.0247 \times 2.1{\times}10^4 = 0.52$ MPa, $N_c = 0.52 \times 74708 = 38.7{\times}10^3$ N

Fusées: $\sigma_c = 0.0247 \times 1.7{\times}10^4 = 0.42$ MPa, $N_c = 0.42 \times 24190 = 10.2{\times}10^3$ N

Time dependent deformation, t = ∞.

Due to shrinkage and creep the load distribution changes. After time t the forces acting on the concrete and the fusées are respectively $N_c - F$ and $N_f + F$. The force F follows from (4.14).

$$F = \frac{(\varepsilon_0\phi + \varepsilon_{rc} - \varepsilon_{rf})\,AE_c}{1 + k\,\phi + AE_c/AE}$$

With: $\varepsilon_0 = 0.0247 \times 10^{-3}$; $\varepsilon_{rf} = 0.10 \times 10^{-3}$; $\varepsilon_{rc} = 0.38 \times 10^{-3}$; $\phi = 4.0$ and $k = 0.5$

$$F = \frac{(0.0247{\times}10^{-3} \times 4.0 + 0.38{\times}10^{-3} - 0.10{\times}10^{-3}) \times 74.708{\times}10^3{\times}2.1{\times}10^4}{1 + 0.5{\times} 4.0 + (74.708 \times 10^3 \times 2.1 \times 10^4)/(24.29 \times 10^3 \times 1.7 \times 10^4)} = 87.4 \times 10^3\ \text{N}$$

After time t the forces acting on the concrete and the fusées are respectively:

$N_c - F = -38.7 + 87.4 = +48.7$ kN $\sigma_c = +0.65$ MPa

$N_f + F = -10.2 - 87.4 = -77.2$ kN $\sigma_f = -3.18$ MPa

Due to the time dependent deformations the fusee's are compressed and the concrete section is tensioned.
Possibly for a not curved structure the joints between fusées don't deform much. Then the forces and stresses due to the time dependent the deformations will be much larger than for a curved structure. The following calculation show the effect of the decreasing deformation of the joints, assuming the fusées are jointed well so these joints do not deform: $\varepsilon_{rf} = 0$. After time t the forces acting on the reinforced concrete and the reinforcement are respectively $N_c - F$ and $N_f + F$ The force F follows from (4.14).

$$F = \frac{(\varepsilon_0 \phi + \varepsilon_{rc} - \varepsilon_{rf})\, AE_c}{1 + k\,\phi + AE_c/AE}$$

With: $\varepsilon_0 = 0.0247 \times 10^{-3}$; $\varepsilon_{rf} = 0$; $\varepsilon_{rc} = 0.38 \times 10^{-3}$; $\phi = 4.0$ and $k = 0.5$

$$F = \frac{(0.0247\times10^{-3} \times 4.0 + 0.38\times10^{-3})\times 74.708\times10^3\times2.1\times10^4}{1 + 0.5\times 4.0 + (74.708 \times 10^3 \times 2.1 \times 10^4)/(24.29 \times 10^3 \times 1.7 \times 10^4)} = 110.5 \times 10^3 \text{ N}$$

After time t the forces acting on the reinforced concrete and the reinforcement are respectively:

$N_c - F = -38.7 + 110.5 = +71.8$ kN $\sigma_c = +0.96$ MPa

$N_f + F = -10.2 - 110.5 = -120.7$ kN $\sigma_f = -4.99$ MPa

Neglecting the deformation of the joints of the fusées due to the shrinkage of the concrete increases the forces and stresses substantially.

Structure composed of concrete, fusées and steel

Assuming a concrete structure is composed of concrete and fusées and reinforced with steel rebars, the structure is subjected to a permanent compressive load N. Due to this load the instantaneous specific deformation is equal to ε_0. Due to the normal force the concrete, fusées and steel are subjected to respectively a force N_c, N_f and N_s. The sum of these three forces is equal to the load:

$$N_c + N_f + N_s = N \qquad [4.15]$$

For a symmetrically loaded structure the specific deformation of the concrete, fusées and reinforcement will be equal, thus:

$$\varepsilon_c = \varepsilon_f = \varepsilon_s = \varepsilon_0. \qquad [4.16]$$

According to Hooke's law the forces in the concrete, fusées and reinforcement follows from respectively: $N_c = AE_c\,\varepsilon_c$; $N_f = AE_f\,\varepsilon_f$; $N_s = AE_s\,\varepsilon_s$

Substituting these expressions into (4.16):

$$\varepsilon_0 = \frac{N_c}{AE_c} = \frac{N_f}{AE_f} = \frac{N_s}{AE_s} \qquad [4.17]$$

Substituting these specific deformations into the expression for the equilibrium of the forces [4.15]:

$$N = AE_c\,\varepsilon_0 + AE_f\,\varepsilon_0 + AE_s\,\varepsilon_0 \qquad [4.18]$$

With this expression we can calculate the instantaneous deformation for t = 0:

$$\varepsilon_0 = \frac{N}{AE_c + A\,E_f + AE_s} \qquad [4.19]$$

The stress in the concrete is: $\sigma_c = E_c\,\varepsilon_c$. Substitution of ε_0 into this expression results in:

$$\sigma_c = E_c\,\varepsilon_c = E_c\,\varepsilon_0 \quad \rightarrow \quad \sigma_c = \frac{E_c\,N}{AE_c[1 + AE_f/AE_c + AE_s/AE_c]} \quad \rightarrow$$

$$\sigma_c = \frac{N}{AE_c\,m_{EA,\,t=0}} \qquad \text{with: } m_{EA,\,t=0} = 1 + AE_f/(A_cE_c) + AE_s/AE_c$$

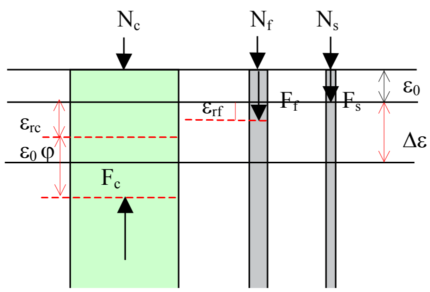

FIGURE 4.4 Time dependent deformations for a structure composed of concrete, steel and fusées

Due to creep the specific deformation will rise with $\phi\,\varepsilon_0$. The total deformation is $\varepsilon_0\,(1+\phi)$. The magnitude of the internal forces follows from the equations describing the equilibrium of the forces and the compatibility. For a concrete structure composed of three materials, concrete, fusées and steel reinforcement, subjected to a normal compressive load the redistribution of the forces due to the time dependent effects will be described. Due to creep and shrinkage the deformation of the structure will increase with $\Delta\varepsilon$. For a time t the specific deformation of the structure is : $\varepsilon_\tau = \varepsilon_0 + \Delta\varepsilon$.

For the concrete the specific deformation is increased due to the creep with $e_0\,\phi$. Due to shrinkage the specific deformation is increased with ε_{rc}. For the fusées the specific deformation will increase due to the deformation of the joints with ε_{rf}. The materials of the composite structure are attached firmly thus the deformations of the three materials must be equal. For a time t the specific deformation of the structure is equal to: $\varepsilon_t = \varepsilon_0 + \Delta\varepsilon$. Due to the deformation of the concrete the fusées and steel must deform too. The steel and fusées are subjected to internal compressive forces respectively F_s and F_f. The concrete is subjected to an internal tensile force F_c. The internal forces are in balance, thus:

$$F_c + F_f + F_s = 0.$$

If the concrete is deformed and the fusées and reinforcement are lengthened then the tensile force in the concrete is compensated by the compressive forces in the fusées and reinforcement:

$$F_c = F_f + F_s\,.$$

Due to the internal force F_c acting on the concrete the specific deformation of the concrete decreased by F_c/AE_c. During the time t the specific deformation due to the force F_c is increases by creep with $F_c\,k\,f/AE_c$. The force F_c is not constant but increasing during the time t, the factor k compensates for the time dependency of this force. Scherpbier showed that this factor is equal to k = ½ [Sch65]. The specific deformation due to the internal force F_c including the creep is:

$$\varepsilon_t = F_c\,(1 + k\,\phi)/AE_c\,.$$

The specific deformation for the concrete is: $\varepsilon_t = \varepsilon_0 + \Delta\varepsilon = \varepsilon_0 + \varepsilon_0\,\phi + \varepsilon_{rc} - \dfrac{F_c(1+k\,\phi)}{AE_c}$ [4.20]

The specific deformation for the fusées is: $\varepsilon_t = \varepsilon_0 + \Delta\varepsilon = \varepsilon_0 + \varepsilon_{rf} + \dfrac{F_f}{AE_f}$ [4.21]

The specific deformation for the rebars is: $\varepsilon_t = \varepsilon_0 + \Delta\varepsilon = \varepsilon_0 + \dfrac{F_s}{AE_s}$ [4.22]

Combining [4.21] and [4.22] results in: $\varepsilon_0 + \varepsilon_{rf} + \dfrac{F_f}{AE_f} = \varepsilon_0 + \dfrac{F_s}{AE_s} \rightarrow F_f = \dfrac{F_s\,AE_f}{AE_s} - \varepsilon_{rf}AE_f$ [4.23]

Combining [4.20] and [4.22] gives: $\varepsilon_0 + \varepsilon_0\,\phi + \varepsilon_{rc} - \dfrac{F_c(1+k\,\phi)}{AE_c} = \varepsilon_0 + \dfrac{F_s}{AE_{fs}} \rightarrow$

$$F_c = -\frac{F_s\,AE_c}{AE_s(1+k\,\phi)} + \frac{(\varepsilon_0\,\phi + \varepsilon_{rc})\,AE_c}{(1+k\,\phi)} \qquad [4.24]$$

Substituting F_f [4.23] and F_c [4.24] into the expression for the equilibrium of the forces: $F_c = F_f + F_s$

$$-\frac{F_s\,A_cE_c}{AE_s(1+k\,\phi)} + \frac{(\varepsilon_0\phi + \varepsilon_{rc})\,AE_c}{(1+k\,\phi)} = \frac{F_s\,AE_f}{AE_s} - \varepsilon_{rf}AE_f + F_s$$

Next the force F_s is calculated with:

$$F_s\left[1 + \frac{AE_f}{AE_s} + \frac{F_s\,AE_c}{AE_s(1+k\,\phi)}\right] = \frac{(\varepsilon_0\phi + \varepsilon_{rc})\,AE_c}{(1+k\,\phi)} + \varepsilon_{rf}AE_f$$

$$F_s = \frac{AE_s\,[(\varepsilon_0\phi + \varepsilon_{rc})\,AE_c/(1+k\,\phi) + \varepsilon_{rf}AE_f\,]}{AE_c/(1+k\,\phi) + AE_f + AE_s} \qquad [4.25]$$

With F_s the force F_f follows from [4.23]: $F_f = \dfrac{F_s\,AE_f}{AE_s} - \varepsilon_{rf}\,AE_f$

The force F_c follows from the equilibrium of the internal forces: $F_c = F_f + F_s$

After the time t the forces acting on the structure composed of concrete, fusées and steel become respectively for the concrete, fusées and reinforcement: $N_c - F_c$; $N_f + F_f$; $N_s + F_s$. Chapter 5 shows for building Q the time dependent effects.

Example: calculation of the stresses including time dependent effects

For the parabolic vault, as described by Van Eck and Bish in Cement [Eck54], the stresses are calculated for loads acting immediately as well as for loads acting during a long time including time dependent effects. The vault has a span of *l* = 24 m and a rise of f = 3.0 m. The thickness of the vault is 200 mm.

Permanent load width 1.0 m:	Dead weight:	2.9 kN/m
	finishing and ceiling:	0.4 kN/m
	Total permanent load:	3.3 kN/m

The stiffness of the concrete and fusées is still calculated with E_f = 21000 MPa. Young's modulus, Area and second moment of the Area for the vault reinforced with distribution bars, width 1.0 m:

Concrete:	E_c = 21000 MPa	$A_c = 89.4 \times 10^3$ mm²	$I_c = 399 \times 10^6$ mm⁴
Fusées:	E_f = 17000 MPa	$A_f = 48.4 \times 10^3$ mm²	$I_f = 128 \times 10^6$ mm⁴
Rebars:	$E_s = 2.1\times10^5$ MPa	A_s = 558 mm²	$I_s = 3.7 \times 10^6$ mm⁴

Forces due to the permanent load

Vertical reactions acting at the supports: $V = ½\, q\, l = ½ \times 3.3 \times 24 = 39.6$ kN

Thrust: $H = \dfrac{q\, l^2}{8\, f} = \dfrac{3.3 \times 24^2}{8 \times 3} = 79.2$ kN

For x = 6.0 m the resulting normal force is: $N = (V^2 + H^2)^{0.5} = 81.6$ kN

Immediately deformation, t = 0, permanent load

Due to the normal force the concrete, fusées and steel are subjected to respectively a force N_c, N_f and N_s. The sum of these three forces is equal to the load: $N_c + N_f + N_s = N$
According to the Theory of Linear Elasticity the forces in the concrete, fusées and reinforcement follows from: $N_c = AE_c\, \varepsilon_c$; $N_f = AE_f\, \varepsilon_f$; $N_s = AE_s\, \varepsilon_s$

Substituting now these forces in the expression of the equilibrium of forces:

$$N = AE_c\, \varepsilon_c + AE_f\, \varepsilon_f + AE_s\, \varepsilon_s$$

For a symmetrical loaded structure the specific deformation of the concrete, fusées and reinforcement will be equal, thus: $\varepsilon_c = \varepsilon_f = \varepsilon_s = \varepsilon_0$.

$$\varepsilon_0 = \frac{N_c}{AE_c} = \frac{N_f}{AE_f} = \frac{N_s}{AE_s}$$

Substitution of the specific deformations in the equation showing the equilibrium of the forces gives fot the specific deformation:

$$\varepsilon_0 = \frac{N}{AE_c + AE_f + AE_s} = \frac{N}{AE_c\, m_{EA\ t=\infty}} \quad \text{with: } m_{EA\ t=\infty} = 1 + AE_f/AE_c + AE_s/AE_c$$

$$m_{EA\ t=\infty} = 1 + \frac{48.4 \times 10^3 \times 17000}{89.4 \times 10^3 \times 21000} + \frac{558 \times 2.1\times10^5}{89.4\times 10^3 \times 21000} = 1.5$$

The specific deformation is: $\varepsilon_0 = \dfrac{81600}{89.4 \times 10^3 \times 2.1 \times 10^4 \times 1.5} = 0.029 \times 10^{-3}$

The normal stress and force in the concrete, fusées is and reinforcement is respectively with $\sigma_x = \varepsilon_0\, E_x$ and $N_x = \sigma_x\, A_x$:

σ_c =	$0.029\times10^{-3} \times 2.1 \times 10^4$ =	0.61 MPa	N_c =	$0.61 \times 89.4 \times 10^3$ =	54.5×10^3 N
σ_f =	$0.029\times10^{-3} \times 1.7 \times 10^4$ =	0.49 MPa	N_f =	$0.49 \times 48.4 \times 10^3$ =	23.7×10^3 N
σ_s =	$0.029\times10^{-3} \times 2.1 \times 10^5$ =	6.09 MPa	N_s =	6.09×558 =	3.4×10^3 N

Instantaneous deformation, t = 0, variable load

Symmetric live load, q = 1.0 kN/m.
Vertical reactions acting at the supports: $V = ½\, q\, l = ½ \times 1.0 \times 24 = 12.0$ kN

Thrust:

$$H = \frac{q\,l^2}{8\,f} = \frac{1.0 \times 24^2}{8 \times 3} = 24.0 \text{ kN}$$

For x = 6.0 m the resulting normal force is: $N = (V^2 + H^2)^{0.5} = 24.73$ kN

According to the Theory of Linear Elasticity the specific deformation follows from:

$$\varepsilon_0 = \frac{N}{AE_c + AE_f + AE_s} = \frac{N}{AE_c\, m_{EA\ t=\infty}} \quad \text{with: } m_{EA\ t=\infty} = 1 + AE_f / AE_c + AE_s / AE_c$$

$$m_{EA\,t=\infty} = 1 + \frac{48.4 \times 10^3 \times 17000}{89.4 \times 10^3 \times 21000} + \frac{558 \times 2.1 \times 10^5}{89.4 \times 10^3 \times 21000} = 1.5$$

Specific deformation is: $\varepsilon_0 = \frac{24730}{89.4 \times 10^3 \times 2.1 \times 10^4 \times 1.5} = 0.00878 \times 10^{-3}$

The normal stress and force in the concrete, fusées and reinforcement is respectively:

σ_c =	$0.00878\times10^{-3}\times 2.1\times10^4$ =	0.184 MPa	N_c =	$0.184\times89.4\times10^3$ =	16.5×10^3 N
σ_f =	$0.00878\times10^{-3}\times 1.7\times10^4$ =	0.15 MPa	N_f =	$0.15\times48.4\times10^3$ =	7.3×10^3 N
σ_s =	$0.00878\times10^{-3}\times 2.1\times10^5$ =	1.85 MPa	N_s =	1.85×558 =	1.0×10^3 N

Instantaneous deformation, t = 0, variable load acting at one side

Asymmetric live load, q = 1.0 kN/m. Vertical reactions acting at the supports:

$V_A = \frac{1}{4} q \times \frac{1}{2} l = \frac{1}{4} \times \frac{1}{2} \times 1.0 \times 24 = 3.0$ kN

$V_B = \frac{3}{4} q \times \frac{1}{2} l = \frac{3}{4} \times \frac{1}{2} \times 1.0 \times 24 = 9.0$ kN

Thrust: $H = \frac{q\,l^2}{16\,f} = \frac{1.0 \times 242}{16 \times 3} = 12.0$ kN

Shear force for x = 6.0 m: $V_{x=6} = \frac{3}{4} q \times \frac{1}{2} l - \frac{1}{4} q \times \frac{1}{2} l = 12.0 - 6.0 = 3.0$ kN

For x = 6.0 m the resulting normal force is : $N = (V^2 + H^2)^{0.5} = 12.369$ kN

According to the Theory of Linear Elasticity the specific deformation is given by:

$$\varepsilon_0 = \frac{N}{AE_c + AE_f + AE_s}$$

The specific deformation is : $\varepsilon_0 = \frac{N}{AE_c\, m_{EA\ t=\infty}}$

Where: $m_{EA\,t=\infty} = 1 + \frac{48.4 \times 10^3 \times 17000}{89.4 \times 10^3 \times 21000} + \frac{558 \times 2.1 \times 10^5}{89.4 \times 10^3 \times 21000} = 1.5$

Specific deformation is: $\varepsilon_0 = \frac{12369}{89.4 \times 10^3 \times 2.1 \times 10^4 \times 1.5} = 0.0044 \times 10^{-3}$

The normal stress and force acting in the concrete, fusées and reinforcement is respectively:

σ_c =	$0.0044 \times 10^{-3} \times 2.1 \times 10^4$ =	0.09 MPa	N_c =	$0.09 \times 89.4 \times 10^3$ =	8.1×10^3 N
σ_f =	$0.0044 \times 10^{-3} \times 1.7 \times 10^4$ =	0.08 MPa	N_f =	$0.08 \times 48.4 \times 10^3$ =	3.9×10^3 N
σ_s =	$0.0044 \times 10^{-3} \times 2.1 \times 10^5$ =	0.92 MPa	N_s =	0.92×558 =	0.5×10^3 N

		symmetric			asymmetric	
		perm. Load	live load	perm. + live load	live load	perm. + live load
Shear force, x = 6.0 m	V	19.8 kN	6 kN	25.8 kN	3 kN	22.8 kN
thrust:	H	79.2 kN	24 kN	103.2 kN	12 kN	91.2 kN
Normal force:	N	81.6 kN	24.7 kN	106.3 kN	12.4 kN	94.0 kN

TABLE 4.4 Resulting forces conform the Theory of Elasticity, for t = 0, x = 6.0 m

			symmetric			asymmetric	
	stresses		perm. Load MPa	live load Mpa	perm. + live MPa	live load MPa	perm. + live load MPa
concrete	normal stress	σ_c	-0.61	-0.18	-0.80	-0.09	-0.70
fusées:	normal stress	σ_f	-0.49	-0.15	-0.64	-0.08	-0.57
reinforcement,:	normal stress	σ_s	-6.09	-1.85	-7.94	-0.92	-7.01
concrete:	bending stress	σ_c			0	±1.78	±1.78

TABLE 4.5 Resulting stresses conform the Theory of Elasticity, for t = 0, x = 6.0 m

Time dependent deformation

Due to the creep and shrinkage the deformation of the structure will increase. The assumption is made that for this vault the specific deformation of the concrete due to shrinkage is about $\varepsilon_{rc} = 0.4 \times 10^{-3}$, and the specific deformation of the fusées due to the deformation of the joints is about $\varepsilon_{rf} = 0.1 \times 10^{-3}$. Further for this vault the creep factor is assumed to be equal to $\phi = 3$. Due to the creep and shrinkage. the normal load acting on a structure composed of fusées, steel and concrete will be redistributed over the components during the life time .The redistribution is calculated with internal forces.The steel and fusées are subjected to internal compressive forces respectively F_s and F_f. and the concrete is subjected to an internal tensile force F_c. for the permanent load the internal force F_s is calculated with [4.25]:

$$F_s = AE_s \frac{[(\varepsilon_0 \phi + \varepsilon_{sc}) AE_c/(1+k\phi) + \varepsilon_{sf} AE_f]}{AE_c/(1+k\phi) + AE_f + AE_s} \qquad [4.25]$$

$$F_s = \frac{558 \times 210000 \times [(0.029 \times 10^{-3} \times 3 + 0.4 \times 10^{-3}) \times 89400 \times 21000/(1+½\times3) + 0.1 \times 10^3 \times 48400 \times 17000]}{89.4 \times 10^3 \times 21000/(1+½\times3) + 48.4 \times 10^3 \times 17000) + 558 \times 2.1 \times 10^5}$$

$$F_s = 31.1 \times 10^3 \text{ N}$$

For the fusées the internal force F_f follows from: $F_f = \frac{F_s A_f E_f}{A_s E_s} - \varepsilon_{rf} A_f E_f \rightarrow$

$$F_f = \frac{31.1 \times 10^3 \times 48.4 \times 10^3 \times 17000}{558 \times 2.1 \times 10^5} - 0.1 \times 10^{-3} \times 48.4 \times 10^3 \times 17000 = 135.7 \times 10^3 \text{ N}$$

The force F_c follows from: $F_c = F_f + F_s \rightarrow F_c = 31.1\times10^3 + 135.7\times10^3 = 166.8 \times 10^3$

After the time t = ∞ the forces and normal stresses acting on the structure composed of concrete, fusées and steel become respectively:

$N_c - F_c =$	$-54.5\times10^3 + 166.8\times10^3 =$	$+\ 12.3 \times 10^3$ N	$\sigma_c =$	+ 1.3 MPa
$N_f + F_f =$	$-27.6\times10^3 - 135.7\times10^3 =$	-163.3×10^3 N	$\sigma_f =$	- 3.4 MPa
$N_s + F_s =$	$-\ 3.4\times10^3 - 31.1\times10^3 =$	$-\ 34.5 \times 10^3$ N	$\sigma_s =$	-61.8 MPa

Possibly the tensile forces acting on the concrete are increased due to varying thermal expansion of the fusées and concrete.

Cracked structure

The concrete is tensioned so it is possible that concrete sections will crack. In a crack the load is transferred by the fusées and reinforcement. The force in the concrete is zero, thus: $N_c - F_c = 0$. The load is transferred by the fusées and the reinforcement:

$$N = (N_c - F_c) + (N_f + F_f) + (N_s + F_s)$$

The specific deformation of the fusées is equal to the specific deformation of the steel, thus:

$$\varepsilon_t = \frac{N_s+F_s}{AE_s} = \varepsilon_{rf} + \frac{N_f+F_f}{AE_f} \rightarrow (N_f + F_f) = AE_f\left(\frac{N_s+F_s}{AE_s} - \varepsilon_{rf}\right)$$

The force $(N_s + F_s)$ follows from the equilibrium of forces:

$$N = AE_f\left(\frac{N_s+F_s}{AE_s} - \varepsilon_{rf}\right) + (N_s + F_s) \rightarrow (N_s + F_s) = \frac{AE_f\,\varepsilon_{rf} + N}{AE_f/AE_s + 1}$$

$$(N_s + F_s) = \frac{48.4\times10^3 \times 1.7\times10^4 \times 0.1\times10^{-3} + 81.6\times10^3}{48.4\times10^3 \times 1.7\times10^4/(558\times2.1\times10^5) + 1} = 20.4 \times 103\text{ N}$$

The force acting at the fusées follows from: $(N_f + F_f) = AE_f\left(\frac{N_s+F_s}{AE_s} - \varepsilon_{rf}\right) \rightarrow$

$$(N_f + F_f) = 48.4\times10^3 \times 2.1\times10^4 \times \left(\frac{20.4\times10^3}{558\times2.1\times10^5} - 0.1\times10^{-3}\right) = 61.0 \times 10^3\text{ N}$$

The force acting on the concrete is:	$N_c = 0$
The force acting on the fusées is:	$(N_f + F_f) = 61.0 \times 10^3$ N
The force acting on the reinforcement is:	$(N_s + F_s) = 20.4 \times 10^3$ N
The normal stress in the concrete is:	$\sigma_c = 0$ MPa
The normal stress in the fusées is:	$\sigma_f = (N_f + F_f)/A_f = -\ 1.3$ MPa
The normal stress in the reinforcement is:	$\sigma_s = (N_s + F_s)/A_s = -36.6$ MPa

Due to the cracking the stresses are decreased. The time dependent values are calculated for a creep factor $\phi = 3$. If the concrete is cracked then the compressive stress is nihil, so the loading acting on the concrete can be assumed as taken away. The concrete deforms by creep if it is subjected to a load, so the creep acts only during the period before the cracking. Thus the effect of the creep is very limited. The resulting forces and stresses (exclusive of second order effects) are given in the following tables.

	Normal force	Internal force	Resulting force	Resulting stress
concrete:	-54.5 kN	+ 166.8 kN	+112.3 kN	+ 1.3 MPa
fusées:	-23.7 kN	− 135.7 kN	- 159.4 kN	3.3 MPa
Reinforcement	-3.4 kN	- 31.1 kN	34.5 kN	-61.8 MPa

TABLE 4.6 Resulting forces due to the permanent load, including time dependent effects t = ∞.

	Resulting force	Resulting stress
concrete:	0 kN	0 MPa
fusées:	-61.0 kN	- 1.3 MPa
reinforcement	-20.4 kN	-36.6 MPa

TABLE 4.7 Forces and stresses due to the permanent load, including time dependent effects t = ∞, in case the vault is cracked.

§ 4.3 Neglecting the fusées, load transfer by concrete and steel

The fusées can transfer the load only in case the fusées are joined properly, it is quite conceivable that due to bad handling the fusées are not joined together and thus cannot transfer any loads. To show the effect of the fusées the forces are calculated in the vault composed of reinforced concrete in case the fusées are not joined properly and cannot transfer the load.

Instantaneous deformation, t = 0

Due to the normal force the concrete, fusées and steel are subjected to respectively a force N_c and N_s. The sum of these three forces is equal to the load: $N_c + N_s = N$ According to the Theory of Linear Elasticity the forces in the concrete, fusées and reinforcement follow from: $N_c = AE_c\,\varepsilon_c$ and $N_s = AE_s\,\varepsilon_s$. Substituting these forces into the expression of the equilibrium of forces: $N = AE_c\,\varepsilon_c + AE_s\,\varepsilon_s$

For a symmetrical loaded structure the specific deformation of the concrete and reinforcement will necessarily be equal, thus: $\varepsilon_c = \varepsilon_s = \varepsilon_0$.

$$\varepsilon_0 = \frac{N_c}{AE_c} = \frac{N_s}{AE_s}$$

Substitution of the specific deformations in the equation of the equilibrium of forces results in:

$$\varepsilon_0 = \frac{N}{AE_c + AE_s} \qquad \varepsilon_0 = \frac{N}{AE_c\, m_{EA\,t=\infty}}$$

$$\text{Where: } m_{EA\,t=\infty} = 1 + \frac{AE_s}{AE_c} = 1 + \frac{558 \times 2.1\times10^5}{89.4 \times 10^3 \times 21000} = 1.06$$

Specific deformation is: $$\varepsilon_0 = \frac{81600}{89.4 \times 10^3 \times 2.1 \times 10^4 \times 1.06} = 0.041 \times 10^{-3}$$

The normal stress in the concrete and reinforcement is respectively:

$\sigma_c = \varepsilon_0 E_c = 0.041 \times 10^{-3} \times 2.1 \times 10^4 = 0.9$ MPa

$\sigma_s = \varepsilon_0 E_s = 0.041 \times 10^{-3} \times 2.1 \times 10^5 = 8.6$ MPa

The forces acting on the concrete and steel are respectively:

$N_c = \sigma_c A_c = 0.9 \times 89.4 \times 10^3 = 76.9 \times 10^3$ N

$N_s = \sigma_s A_s = 8.6 \times 558 = 4.8 \times 10^3$ N

Time dependent effect t = ∞

Due to creep and shrinkage the deformation of the structure will increase with $\Delta\varepsilon$. At time t = ∞ the specific deformation of the structure is equal to: $\varepsilon_{t=¥} = \varepsilon_0 + \Delta\varepsilon$. Due to shrinkage the concrete deforms. The specific deformation of the concrete due to the shrinking is ε_{rc}. For this structure the assumption is made that the specific deformation of the concrete due to shrinkage is $\varepsilon_{rc} = 0.4 \times 10^{-3}$.
Due to creep a concrete structure subjected to a compressive load will deform during its life time with $\phi\, \varepsilon_0$. The total deformation of the concrete is equal to $\varepsilon_0 (1+\phi)$. For this concrete structure in compression the creep factor is assumed to be equal to $\phi = 3$. Due to the deformation of the concrete the steel must necessarily deform also. The steel is subjected to internal compressive forces F_s and the concrete is subjected to an internal tensile force F_c. The internal forces are in balance, thus: $F_c = F_s = F$
The internal force F is calculated with [4.6]:

$$F = \frac{AE_c(\varepsilon_0 \phi + \varepsilon_{rc})}{(1 + k\phi + (AE_c)/AE_s}$$

$$F = \frac{(89.4 \times 10^3 \times 21000) \times (0.041 \times 10^{-3} \times 3 + 0.4 \times 10^{-3})}{1 + ½ \times 3 + (89.4 \times 10^3 \times 21000)/(558 \times 2.1 \times 10^5)} = 53.0 \times 10^3 \text{ N}$$

The force acting on the concrete and reinforcement is respectively:

$N_c - F = -76.9 \times 10^3 + 53.0 \times 10^3 = -23.9 \times 10^3$ N

$N_s + F_s = -4.8 \times 10^3 - 53.0 \times 10^3 = -57.8 \times 10^3$ N

The normal stress in the concrete is: $\sigma_c = N_c/A_c = -0.3$ MPa

The normal stress in the reinforcement is : $\sigma_s = N_s/A_s = -103.6$ MPa

The concrete and steel are both compressed. For this structure it seems preferable to join the fusées badly so that the loads are transferred only by the steel and the concrete. Thus the concrete is compressed by the permanent load. The stiffness of the vault is not reduced by cracks caused by the redisribution of the load. The concrete is tensioned and perhaps cracked only due to bending moments caused by asymmetric loads.

§ 4.4 Time dependent effects and increasing buckling risk

A structure subjected to a normal load can fail due to the buckling. Assume a column is subjected to a compressive normal load N acting centrically. The column is supported at the ends by hinges. The column is not completely straight but slightly curved. Halfway up the column the deformation is equal to u_0. Due to the normal load and the curvature the column is subjected to bending moments. The maximum moment acting halfway is equal to $M_0 = N\, u_0$. The column will deform due to this bending moment with u_0/n_{cr}.

Due to the deformation u_0/n_{cr} the bending moment increases with: $M_0/n_{cr} = N u_0/n_{cr}$.
The column will deform due to this bending moment with u_0/n_{cr}^2. Due to the deformation u_0/n_{cr}^2 the bending moment increases with $M_0/n^2 = N u_0/n_{cr}^2$, and so on.
The total deformation follows from: $u_{tot} = u_0 + u_0/n_{cr} + u_0/n_{cr}^2 ... = u_0 + u_0/(n_{cr}-1)$

The total bending moment now becomes: $M_t = n_{cr} M_0/(n_{cr} - 1)$

Due to creep the deformation will increase. Assuming the increase of the deformation due to the creep is equal to Δu_∞. The bending moment increases to $N \Delta u_\infty$. The increase of deformation follows from:

$$\Delta u_\infty = \frac{\phi u_0}{(n_{cr} - 1)} + \frac{N \Delta u_\infty}{M_t} \times \frac{u_0 (1 + k \phi)}{(n_{cr} - 1)}$$

Substituting $M_t = \frac{N n_{cr} u_0}{n_{cr} - 1}$: $\Delta u_\infty \quad \frac{\phi u_0}{(n_{cr} - 1)} + \frac{(1 + k\phi) N \Delta u_\infty u_0}{N n_{cr} u_0} \rightarrow$

$$\Delta u_\infty - \frac{\Delta u_\infty (1 + k\phi)}{n_{cr}} = \frac{\phi u_0}{(n_{cr} - 1)} \rightarrow \Delta u_\infty = \frac{n_{cr} \phi u_0}{(n_{cr} - 1)(n - k\phi - 1}$$

The total bending moment is: $M_{t\infty} = \frac{N u_0 n_{cr}}{n_{cr} - 1} + N \Delta u_\infty \rightarrow$

$$M_{t\infty} = \frac{n_{cr}}{(n_{cr} - 1)} \times \frac{N u_0}{(n_{cr} - 1)} + \frac{n_{cr} N u_0 \phi}{(n - k\phi - 1)} \rightarrow M_{t\infty} = \frac{N u_0 n_{cr}}{(n_{cr} - 1)} \times \frac{(n_{cr} - k\phi - 1 + \phi)}{(n_{cr} - k\phi - 1}$$

Due to the creep the bending moment is increased with the factor:

$$\frac{n_{cr} - 1 - k\phi + \phi}{n_{cr} - 1 - k\phi} \qquad [4.26]$$

For k = ½ the factor becomes: $\frac{n_{cr} - 1 + ½\phi}{n_{cr} - 1 - ½\phi}$

For $n_{cr} = 5$ en $\phi = 3$ the factor becomes $^{11}/_5$, $M_t = M_0 \times \frac{5}{(5 - 1)} \times {}^{11}/_5 = 2.75 M_0$

Approach

In practice the increase of the second order effects is approximated often by reducing the stiffness EI_0 with a factor $(1 + \phi)$, so: $EI_\infty = EI_0/(1 + \phi)$.
Due to the reduction of the stiffness the buckling ratio n is decreased with: $n_{cr\infty} = n_{cr}/(1 + \phi)$.
The bending moment including creep effects is: $M_{t\infty} = N u_0 n_{cr\infty}/(n_{cr\infty} - 1)$
Substituting $n_{cr\infty} = n_{cr}/(1+ \phi)$: $M_{t\infty} = \frac{N u_0 n_{cr}/(1 + \phi)}{n_{cr}/(1 + \phi) - 1}$

Multiply counter and nominator with $(1+ \phi)$: $M_{t\infty} = \frac{N u_0 \; n_{cr}}{(n_{cr} - 1 - \phi)}$

Due to second order effect and creep the bending moment M_0 increases with the factor $n_{cr}/(n_{cr} -1 - \phi)$
For $n_{cr} = 5$ and $\phi = 3$ the bending moment becomes: $M_t = 5 M_0 > 2.75 M_0$

Comparing the values with the values found for the former calculation shows that the result is negative especially for larger values of ϕ. The approach is on the safe side.

Example including second order effects

For the parabolic vault, as described by Van Eck and Bish in Cement [Eck54], the second order effect is calculated for loads acting immediately as well as for loads acting during a long time including time dependent effects.

t = 0, Young's modulus, Area and second moment of the Area:

$E_c = 2.1 \times 10^4$ MPa	$A_c = 89.4 \times 10^3$ mm^2	$I_c = 399 \times 10^6$ mm^4
$E_f = 1.7 \times 10^4$ MPa	$A_f = 48.4 \times 10^3$ mm^2	$I_f = 128 \times 10^6$ mm^4
$E_s = 21 \times 10^4$ MPa	$A_s = 558$ mm^2	$I_s = 3.7 \times 10^6$ mm^4

The stiffness of the structure is defined with: $EI_{t=\infty} = EI_c + EI_f + EI_s$

This expression can be rewritten into: $EI_{t=\infty} = EI_c\, m_{EI\ t=\infty\ w}$ with: $m_{EI\ t=\infty} = 1 + EI_f/EI_c + EI_s/EI_c$

Substituting the second moment of the area and Young's modulus for concrete, fusées and reinforcement gives:

$$m_{EI\ t=\infty} = 1 + \frac{17000 \times 128 \times 10^6}{21000 \times 399 \times 10^6} + \frac{2.1 \times 10^5 \times 3.7 \times 10^6}{21000 \times 399 \times 10^6} = 1.35$$

The stiffness of the structure is: $EI_{t=\infty} = EI_c \times 1.35 = 21000 \times 399 \times 10^6 \times 1.35 = 11.3 \times 10^{12}$ Nmm^2

For a vault the buckling force is defined conform the expression defined by Timoshenko, see [Tim52]:

$$N_{cr} = \frac{EI\,[\pi^2/\phi^2 - 1]}{R^2} \qquad [2.6]$$

The radius is equal to R = 26.8 m and the angle is equal to $\phi = 0.46$ radians. Substituting these values into expression [2.6] to define the buckling force results in:

$$N_{cr} = \frac{11.3 \times 10^{12}\,[\pi^2/0.46^2 - 1]}{26.8^2 \times 10^6} = 718 \times 10^3 \text{ N}$$

Asymmetrical loading, including creep

Loads:	dead load:	2.9 kN/m^2
	finishing and ceiling:	0.4 kN/m^2
	total permanent load:	3.3 kN/m^2
	asymmetrical live load:	1.0 kN/m^2

Due to the permanent load and asymmetrical live load the vault is subjected to the following forces and bending moments.

Vertical reactions: $V_A = ½\, q_g \times l + ¼\, q_e \times ½\, l = ½ \times 3.3 \times 24 + ¼\, 1.0 \times ½ \times 24 = 42.6$ kN/m

$V_B = ½\, q_g \times l + ¾\, q_e \times ½\, l = ½ \times 3.3 \times 24 + ¾\, 1.0 \times ½ \times 24 = 48.6$ kN/m

Thrust: $$H = \frac{q_g\, l^2}{8f} + \frac{q_e\, l^2}{16 \times f} = \frac{3.3 \times 24^2}{8 \times 3} + \frac{1.0 \times 24^2}{16 \times 3} = 91.2 \text{ kN/m}$$

The bending moment is at maximum for x = ¼ l, the normal force at this position is :

$V = 48.6 - 4.3 \times ½ \times 12 = 22.8$ kN, $H = 91.2$ kN

$N = (V^2 + H^2)^{0.5} = 94$ kN

Bending moment: $$M_o = \frac{q_e\, l^2}{64} = \frac{1.0 \times 24^2}{64} = 9.0 \text{ kNm}$$

Due to the second order effects the bending moment will increase: $M = M_o + \frac{N \Delta n}{n - 1}$

The curvature of the bending moment is taken as sinusoidal. Using this assumption the deformation of the vault due to the bending moment can be calculated with:

$$\Delta = \frac{M_o l^2}{EI \times 4 \times \pi^2}$$

Substitute EI, l and M_o: $\Delta = \frac{M_o l^2}{EI \times 4 \times \pi^2}$ $\rightarrow$ $\Delta = \frac{9 \times 10^6 \times 24^2 \times 10^6}{11.3 \times 10^{12} \times 4 \times \pi^2} = 11.6 \text{ mm}$

The ratio of the buckling and normal force is: $n_{cr} = N_{cr}/N = 718/94 = 7.6$

The bending moment including second order effects is:

$$M_t = M_o + \frac{N \Delta n}{n - 1} = 9.0 \times 10^6 + \frac{94 \times 10^3 \times 11.6 \times 7.6}{7.6 - 1} = 10.3 \times 10^6 \text{ Nmm}$$

In practice the bending moment including second order effects is often calculated with:

$$M_t = \frac{M_o n_{cr}}{n_{cr} - 1} = \frac{9.0 \times 10^6 \times 7.6}{7.6 - 1} = 10.4 \times 10^6 \text{ Nmm}$$

This result is only slightly larger than the result calculated before.

Due to the creep the deformations increase. The increase of the deformation can be considered as a decrease of the stiffness of the structure. According to expression [4.26] due to the creep the bending moment is increased with the factor:

$$(n_{cr} - 1 - k\phi + \phi)/(n_{cr} - 1 - k\phi) \qquad [4.26]$$

For k = ½ the factor becomes: $(n_{cr} - 1 + \frac{1}{2}\phi)/(n_{cr} - 1 - \frac{1}{2}\phi)$

With $\phi = 3$ the bending moment increases with a factor: $\frac{7.6 - 1 + \frac{1}{2} \times 3}{7.6 - 1 - \frac{1}{2} \times 3} = 1.58$

The bending moment inclusive this increase is: $M_{t=\infty} = 10.3 \times 1.58 = 16.3 \text{ kNm}$

Conforming to the theory of elasticity the bending stress in the concrete is calculated with:

$$\sigma_c = \frac{M_{t¥} \frac{1}{2} t E_c}{EI_o} \rightarrow \sigma_c = \pm \frac{16.3 \times 10^6 \times \frac{1}{2} \times 200 \times 21000}{11.3 \times 10^{12}} = \pm 3.0 \text{ MPa}$$

Due to creep the bending stresses acting on the concrete increases considerately.

Approach

In practice the increase of the second order effects is approximated often by reducing the stiffness EI_0 with a factor $(1 + \phi)$, so: $EI_\infty = EI_0/(1 + \phi)$. Due to the reduction of the stiffness the buckling ratio n_{cr} is decreased with: $n_{cr\infty} = n_{cr}/(1 + \phi)$.

The bending moment including creep effects is: $M_{t=\infty} = N u_0 n_{cr\infty}/(n_{cr\infty} - 1)$

Substituting $n_{cr\infty} = n_{cr}/(1 + \phi) = 7.6/(1+3)$ and $N u_0 = 9$ kNm: $M_{t=\infty} = \frac{9 \times 7.6/4}{7.6/4 - 1} = 19.0 \text{ kNm}$

Comparing this values with the value found for the former calculation shows that the result of the approach is on the safe side.

§ 4.5 Bending moments acting on a structure composed of concrete, fusées and steel

Assume a structure composed of concrete and fusées and reinforced with steel rebars is subjected to a bending moment M. This bending moment is resisted by the concrete, fusées and steel so these elements are subjected to respectively a bending moment M_c, M_f and M_s. Due to the creep the load transfer is changed. Firstly the instantaneous moments are defined next the effect of creep is analysed. Instantaneous moments. Due to this moment the concrete, fusées and steel are subjected to respectively a bending M_c, M_f and M_s. The sum of these three moments is the bending moment M:

$$M_c + M_f + M_s = M \quad [4.27]$$

For a symmetrical loaded structure the curvature of the concrete, fusées and reinforcement will be equal, thus:

$$\kappa = M_c/EI_c = M_f/EI_f = M_s/EI_s \quad [4.28]$$

With this equation the bending moment acting on the fusées and rebars can be expressed as follows:

$$M_f = M_c\, EI_f/EI_c \quad [4.29]$$

$$M_s = M_c\, EI_s/EI_c \quad [4.30]$$

Substituting these specific deformations into the expression for the equilibrium of the bending moments results in the following expression:

$$M = M_c + M_c\, EI_f/EI_c + M_c\, EI_s/EI_c \quad [4.27']$$

With this expression we can calculate the bending moment acting on the concrete at t = 0:

$$M_c = M/m_{EI}\text{, with: } m_{EI} = [1 + EI_F/EI_c + EI_s/EI_c] \quad [4.31]$$

Next the bending moments acting on the fusées M_f and rebars M_s are calculated with [4.29] and [4.30].

Time dependent effects

Due to creep the specific deformation will increase with $\phi\, \varepsilon_0$. The total deformation is $\varepsilon_0\,(1 + \phi)$. The magnitude of the internal forces follows from the equations describing the equilibrium of the forces and the compatibility. For a concrete structure composed of three materials, concrete, fusées and steel reinforcement, subjected to a normal compressive load the redistribution of the forces due to the time dependent effects will be described.

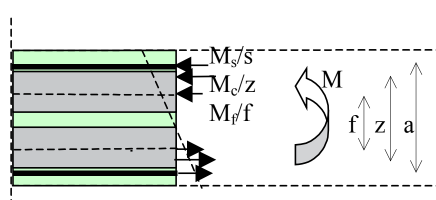

FIGURE 4.5 Bending moments acting on a element composed of fusées, concrete and rebars.

For the concrete the curvature is increased due to creep with $M_c\,\phi/EI_c$. The materials of the composite structure are attached firmly thus the deformations of the three materials must be equal. For a time t the curvature of the structure is equal to K_t. Due to the curvature of the concrete the fusées and steel must also deform. The steel and fusées are subjected to internal bending moment U_s and U_f. The concrete is subjected to an internal moment U_c. The internal moments are in equilibrium, thus:

$$U_c = U_f + U_s \qquad [4.32]$$

Due to the internal moment U_c acting on the concrete the curvature of the concrete is decreased with: U_c/E_cI_c.
During the time t the curvature due to this moment M_c is increased by creep with $k \times U_c \times \phi/E_cI_c$.
The moment M_c is not constant, but increases over time t, the factor k compensates for the time dependency of this force, Scherpbier has shown that this factor is approximately equal to k = ½ [Sch65]. The curvature of the concrete, fusées and steel rebars are equal:

$$\kappa_t = M_c\phi/EI_c - U_c(1+k\phi)/EI_c = U_f/EI_f = U_s/EI_s \qquad [4.33]$$

Thus: [4.33']

$$U_F = [M_c\,\phi - U_c[1+k\phi)]\,EI_f/EI_c$$

$$U_s = [M_c\,\phi - U_c(1+k\phi)]\,EI_s/EI_c \qquad [4.33']$$

Substituting U_F and U_s into the equilibrium of internal moments [4.32]:

$$U_c = [M_c\phi - U_c(1+k\phi)]\,EI_f/EI_c + [M_c\phi - U_c(1+k\phi)]\,EI_s/EI_c$$

$$U_c[1 + (1+k\phi)\,EI_f/EI_c + (1+k\phi)\,EI_s/EI_c] = M_c\,\phi\,(EI_F + EI_s)/EI_c$$

$$U_c = \frac{M_c\phi\,(EI_f + EI_s)/(1+k\phi)}{EI_c/(1+k\phi) + EI_f + EI_s} \qquad [4.34]$$

Next the internal moments acting on the fusées and rebars are calculated with [4.33'] and [4.33'].

$$U_F = M_c\phi\,m\,EI_f/EI_c \text{ and } U_s = M_c\,\phi\,m\,EI_s/EI_c \quad \text{with: } m = \frac{EI_c/(1+k\phi)}{EI_c/(1+k\phi) + EI_f + EI_s}$$

After the time t the bending moments acting on the concrete, fusées and steel become respectively:

concrete:	$M_c - U_c$
fusées:	$M_f + U_f$
reinforcement :	$M_s + U_s$

Example bending moments acting on a structure composed of concrete, fusées and steel

Assuming a structure composed of concrete and fusées and reinforced with rebars is subjected to a bending moment M, this bending moment is resisted by the concrete, fusées and steel. Thus these elements are subjected to respectively a bending moment M_c, M_f and M_s. Due to creep the load transfer is changed. Firstly the instantaneous moments are defined, next the effects of creep are analysed.

Instantaneous moments

Due to this moment the concrete, fusées and steel are subjected to respectively a bending moment M_c, M_f and M_s. Assume the structure is subjected to a moment M = 1.0 kNm = 10^6 Nmm. The sum of these three moments is equal to the bending moment M:

$$M_c + M_f + M_s = M \qquad [4.27]$$

The stiffness of the concrete, fusées and rebars is:

$EI_c = 399 \times 10^6 \times 21000 = 8.37 \times 10^{12}$ Nmm².

$EI_F = 128 \times 10^6 \times 17000 = 2.18 \times 10^{12}$ Nmm².

$EI_s = 3.7 \times 10^6 \times 210000 = 0.78 \times 10^{12}$ Nmm².

For a symmetrically loaded structure the curvature of the concrete, fusées and reinforcement will be equal, thus:

$$\kappa = M_c/EI_c = M_f/EI_F = M_s/EI_s \quad [4.28]$$

With this equation the bending moment acting on the fusées and rebars can be expressed as follows:

$$M_f = M_c\, EI_F/EI_c \quad [4.29]$$

$$M_s = M_c\, EI_s/EI_c \quad [4.30]$$

Substituting these specific deformations into the expression for the equilibrium of the bending moments [4.27]:

$$M_c = M/m_{EI} \quad \text{with: } m_{EI} = 1 + (EI_F + EI_s)/EI_c = 1 + (2.18 + 0.78)/8.37 = 1.35$$

$$M_c = M/m_{EI} = M/1.35 = 0.74 \text{ kNm}$$

Next the bending moments acting on the fusées M_f and rebars M_s are calculated with respectively [4.29] and [4.20]:

$$M_f = M_c\, EI_F/EI_c = 0.74 \times 2.18/8.37 = 0.19 \text{ kNm}$$

$$M_s = M_c\, EI_s/EI_c = 0.74 \times 0.78/8.37 = 0.07 \text{ kNm}$$

Time dependent effects

Due to the creep the specific deformation will rise with $\phi\, \varepsilon_0$. The total deformation is $\varepsilon_0 (1+\phi)$. For the concrete structure the curvature is increased due to the creep with $M_c\, \phi/EI_c$. The materials of the composite structure are attached firmly thus the deformations of the three materials must necessarily be equal. At time t the curvature of the structure is equal to k_t. Due to the curvature of the concrete the fusées and steel must also deform. The steel and fusées are subjected to internal bending moments U_s and U_f. The concrete is subjected to an internal moment U_c. The internal moments are in equilibrium, thus:

$$U_c = U_f + U_s \quad [4.32]$$

Due to the internal moment U_c acting on the concrete the curvature of the concrete is decreased with: U_c/EI_c. During the time t the curvature caused by moment M_c is increased by creep with $k\, U_c \phi/EI_c$. The moment M_c is thus not constant, but increases during the time t, the factor k compensates for the time dependency of this force, Scherpbier showed that this factor is approximately equal to k = ½ [Sch65]. For this structure we assume for the creep ϕ = 3. The intermal bending moment actting on the concrete follows from equation (4.33):

$$U_c = \frac{M_c \phi\, (EI_F + EI_s)/(1 + k\,\phi)}{EI_c/(1 + k\,\phi) + EI_F + EI_s}$$

$$U_c = \frac{0.74 \times 3 \times (2.18 + 0.78) \times 10^{12}/(1 + ½ \times 3)}{8.37 \times 10^{12}/(1 + ½ \times 3) + 2.18 \times 10^{12} + 0.78 \times 10^{12}} = 0.42 \text{ kNm}$$

Next the internal moments acting on the fusées and rebars are calculated with [4.32'] and [4.32''].

$$U_F = M_c \phi\ m\, EI_F/EI_c$$

with: $m = \dfrac{EI_c/(1+k\,\phi)}{EI_c/(1+k\,\phi) + EI_F + EI_s} = \dfrac{8.37\times10^{12}/(1+½\times3)}{8.37\times10^{12}/(1+{}^1/_2\times3) + 2.18\times10^{12} + 0.78\times10^{12}} = 0.53$

$U_F = M_c\,\phi\,m\,EI_F/EI_c = 0.74\times3\times0.53\times2.18/8.37 = 0.30$ kNm
$U_s = M_c\,\phi\,m\,EI_s/EI_c = 0.74\times3\times0.53\times0.78/8.37 = 0.11$ kNm

At time t the bending moments acting on the concrete, fusées and steel become respectively:

concrete:	$M_c - U_c = 0.74 - 0.42 = 0.32$ kNm
fusées:	$M_f + U_f = 0.19 + 0.30 = 0.49$ kNm
reinforcement :	$M_s + U_s = 0.07 + 0.11 = 0.18$ kNm

Due to creep the bending moment acting on the concrete is reduced considerately.

§ 4.6 Approach of the factor k

The previous chapters explain the effects of time dependent material behaviour on composite structures. Structures of concrete deform immediately after being loaded. But after some time the deformations increase due to time dependent effects such as creep and shrinkage. For composites the time dependent deformations of concrete parts are prevented partly by composites as steel which do not deform or deform less than the concrete parts by time dependent effects. Due to the time dependent effects the loads are redistributed. For the composite structure the redistribution is defined by introducing internal forces acting on the parts. The internal forces are not constant but increase during the time. The deformation due to the internal force acting on the concrete is also increased by creep, with a factor k ϕ.

Scherpbier gives the following analysis to approach the factor k [Sch65].
Assume a column is subjected to a normal force N. Due to this load the column will deform and the strain is: $\varepsilon_0 = N/EA$. Next the column is fixed. Due to the creep the column will deform with: $\varepsilon_0\,\phi$.
The column is fixed so the deformation due to creep is prevented. To prevent creep deformation the column is subjected to a force F_t. This force is not constant but increases during the time. Due to this force F_t the column will deform:

$$\varepsilon_t = \frac{F_t\,k\,\phi_t}{EA} \qquad [4.34]$$

The column is fixed, the deformation due to the creep is prevented by the force F_t , thus:

$$\frac{\phi_t\,N}{EA} - \frac{F_t\,k\,\phi_t}{EA} = 0 \qquad [4.35]$$

The internal force F_t increases during a long interval. If t = ∞ , the force is equal to F_∞. The force is related to the deformation so: $F_t = F_\infty\,\phi_t/\phi_\infty$

At time t the deformation changes, so the strain is: $d\varepsilon_t = \dfrac{F_t\,\phi_t d\phi_t}{EA} = \dfrac{F_\infty\,\phi_t d\phi_t/\phi_\infty}{EA}$

Integrating the strain gives: $\int d\varepsilon_t = \int \dfrac{F_\infty\,\phi_t d\phi_\tau}{EA\,\phi_\infty} \approx \dfrac{½\,F_\infty\phi_\tau^{\,2}}{EA\,\phi_\infty} + C$

For t = 0: $F_t = 0$ and $d\phi_t = 0$, so C = 0: $\varepsilon_t = \dfrac{½\,F_\infty\,\phi_\tau^{\,2}}{EA\,\phi_\infty}$ [4.36]

Firstly we assumed the strain was [4.34]: $\varepsilon_t = \dfrac{k F_t \phi_t}{EA}$

Substituting $F_t = F_\infty \phi_t / \phi_\infty$ $\varepsilon_{t} = \dfrac{k F_\infty \phi_t^2}{EA \phi_\infty}$ [4.37]

The strain calculated with expression [4.36] is equal to the strain calculated with [4.37] so k = ½. Actually this value approaches k, Scherpbier writes that k = ½ is often a very good approach. The factor k can be defined more accurate as follows for an internal force F_t increasing in time.

The factor k defined for an increasing force F_t

Scherpbier describes the following analysis to define the factor k accurately [Sch65]. Assume a column is subjected to a normal force N. Due to this load the column will deform, the strain is: $\varepsilon_0 = N/EA$. Due to creep the column will deform with $\varepsilon_0 \phi$. The column is fixed so the deformation due to the creep is prevented. To prevent creep deformation the column is subjected to a force F_t. This force is not constant but increasing during the time t. At a certain moment the strain due to the internal force F_t is:

$$\varepsilon_t = \frac{F_t d\phi_t + dF_t}{EA} \quad [4.38]$$

The column is fixed, the deformation due to the creep is prevented by the force F_t thus:

$$\frac{N d\phi_t}{EA} - \frac{F_t d\phi_t + dF_t}{EA} = 0 \quad [4.39]$$

Assuming the internal force F_t is equal to $p_t N$ and substituting this value into [4.39]:

$$\frac{N d\phi_t}{EA} - \frac{p_t N d\phi_t + N dp_t}{EA} = 0 \rightarrow (1 - p_t)\, d\phi_t + dp_t = 0 \rightarrow$$

$$d\phi_t = - \frac{dp_t}{(1 - p_t)} \quad [4.40]$$

Integrating [4.40] gives: $\int d\phi_t = - \int \dfrac{dp_t}{(1 - p_t)} \rightarrow \phi_t = - \ln(1 - p_t) + C$

For t = 0: $\phi_t = 0$, $F_t = 0$ so also $p_t = 0$, so C = 0: $\phi_t = - \ln(1 - p_t)$ [4.41]

The increase of the strain $d\varepsilon_t$ due to $F_t = p_t N$ is: $d\varepsilon_t = \dfrac{p_t N d\phi_t}{EA}$ [4.42]

Substituting [4.41] into [4.42]: $d\varepsilon_t = \dfrac{p_t N dp_t}{EA (1 - p_t)}$

This expression can be written as: $d\varepsilon_t = \dfrac{N (1 - 1 + p_t) N dp_t}{EA (1 - p_t)}$

This expression can be written as: $d\varepsilon_t = (-1 + \dfrac{1}{(1 - p_t)}) \times \dfrac{N dp_t}{EA}$ [4.43]

Integrating [4.43] gives: $\varepsilon_t = \dfrac{N [-p_t - \ln(1 - p_t)]}{EA} + C$

For t = 0: $\varepsilon_t = 0$, $F_t = 0$ so also $p_t = 0$, so C = 0: $\varepsilon_t = \dfrac{N [- p_t - \ln(1 - p_t)]}{EA}$ [4.44]

With the factor k the specific deformation is: $\varepsilon_t = \dfrac{N p_t k \phi_t}{EA}$ [4.34]

Substituting $\phi_t = - \ln(1 - p_t)$ [4.41] into [4.34] gives: $\varepsilon_t = - \dfrac{N p_t k \ln(1 - p_t)}{EA}$ [4.45]

The strain expressed with equation [4.44] is equal to the strain expressed with equation [4.45]. The factor k follows from:

$$-k\,p_t \ln(1 - p_t) = -p_t - \ln(1 - p_t) \qquad \rightarrow \qquad k = 1/p_t + 1/\ln(1 - p_t)$$

Thus the factor k is depending on the ratio $p_t = F_t/N$. The following table shows k for F_t/N varying from 0.2 to 0.8.

$P_t = F_t/N$	K
0.2	0.52
0.4	0.54
0.6	0.57
0.8	0.63

TABLE 4.8 The ratio k and p_t

Table 4.8 shows the factor k for several values of $p_t = F_t/N$. The factor k is slightly larger than k = ½. Otherwise the creep is not constant but will decrease in time. So if a load is acting at a time $t = t_2$ then the creep factor $\phi_{t=t2}$ is smaller than the creep factor $\phi_{t=t1}$ for a force acting at time $t = t_1$. Consequently Scherpbier asserts that including the increasing creep with k = ½ will be a safe engineering approach [Sch65].

§ 4.7 Conclusions

For a stucture composed of concrete, fusees and reinforcement the effect of the shrinkage and creep is significant. For the vault subjected to a compressive normal force, creep and shrinkage of the concrete will change the load transfer. Due to the time dependent deformations the normal force acting at the concrete covering the rebars and fusées decreases and the normal forces acting at the reinforcement and fusées increase. Possibly due to the redistribution of the load the concrete is tensioned. The concrete will crack if the tensile stress exceeds the maximum tensile stress. For the cracked zones the stress in the concrete is zero, the normal forces are transferred by the reinforcement and the fusées only. Thus in a cracked section the internal forces are changed, due to the decrease of the concrete tensile stress the compressive stresses in the fusées and steel are reduced too.
According to the load transfer by the fusées and reinforcement the vault can be considered as an arch of ceramic elements joined with a reinforcement to resist bending moments due to asymmetric live loads.
The internal forces are not constant but increasing during the time. Consequently the deformation due to these forces increases too. The deformation of the concrete caused by the internal force increases due to creep. This deformation is reduced with a factor k to compensate for the time dependency of the internal force acting at the concrete. Scherpbier showed that this factor is equal to k = ½ [Sch65].
The stiffness of the structure will be reduced highly by the tensile forces acting on the concrete due to the time dependent deformations. Probably the tensile forces acting on the concrete are increased due to varying thermal expansion of the fusées and concrete. Especially for cracked structures the stiffness will be reduced substantially. Consequently the buckling resistance will be lower. Possibly due to the change of the load transfer, caused by the time dependent deformations, these structures can become unsafe. Chapter 5 shows for a fusée vault the effect of the time dependent deformations and the reduction of the stiffness.

A solution for the reduction of the load transfer by the concrete could be an increase of the ductility of the infill. Reducing the stiffness of the infill will reduce the load transfer by the infill, and reduce the redistribution of the forces due to the time dependent deformations. Thus the loads will be transferred mostly by the concrete and steel rebars. Concerning the stiffness cardboard tubes will be a better infill than ceramic elements. Chapter 8 compares and discusses several infill elements.

5 Analysis of the Fusée Céramique roof of building Q in Woerden

In 1955 two buildings were constructed in Woerden near the medieval castle at the so called 'Defensieterrein' for the Dutch army. A storage warehouse, labelled building Q, and a workshop for constructing and repairing tents, labelled building R. Both buildings were roofed with a Fusée Céramique vault.
Building Q had a rectangular plan with a length of 85.745 m and a width of 20.24 m. The height of the gutter and the height to the top of the curved roof was respectively 4.92 m and 7.555 m. The thickness of the roof of building Q is only 130 mm, much thinner than advised by Van Eck and Bish [Eck54]. The span of the roof is 19.8 m. The ratio of thickness to span is very low at: t/l =130/19800 m = 1/152. The roof lights, centre to centre 15.015 m, provide the roof with expansion joints and separate the vault into independent parts.

FIGURE 5.1 Front of the workshop labelled building Q, former Military complex, Woerden, The Netherlands

FIGURE 5.2 Demolishing building Q in 2012.

Building R was smaller than building Q and was composed of three distinct parts. The centre of the workshop had a length of 41.9 m and a width of 17.04 m. The height of the gutter was 4.52 m. At the front and rear the building was extended with an annex. A small annex was made at the front, with a length of 4.94 m and a width of 13.08 m. The height of the gutter of the roof was 3.25 m. A larger annex was made at the rear, with a length of 27.05 m and a width of 10 m. The height of the gutter of the roof was again 4.52 m.
Both buildings used cylindrical piles with a length of 3.0 m as foundation, starting from the concrete beams at – 1.5 m till – 4.5 m below the floor. Two diameters were used, piles with an inner diameter of 1.0 m and thickness of 100 mm and piles with an inner diameter of 1.15 m and a thickness of 110 mm. Building Q and R were pulled down in 2012. The redundancy of these buildings as demonstrated during demolition was amazing, heavy blows were needed to demolish the Fusée Céeramique roofs. To understand the amazing load bearing capacity of these roofs an analysis of the bearing capacity of building Q is made. The calculation is reconstructed conform the methods in usage in the Netherlands during the nineteen fifties. In 1955 Van Eck and Bish wrote an article in Cement [Eck54] describing the structural design calculations for a fusée roof. According to this analysis and the calculations made for a workshop in Dongen a reconstruction of the design and calculations of the fusée roof of building Q is made. Firstly the geometry and the features of the materials are defined. Next the forces, bending moments and stresses are calculated. Finally the calculations are validated using current methods.

§ 5.1 Geometry and loads

The parabolic vault had a span of *l* = 19.8 m centre to centre, half of the span is equal to a = 9.9 m. The top of the roof was at 7.555 m above the ground floor. The bottom of the gutter was positioned at a height of 4.92 m. The thickness of the roof and gutter was respectively 130 mm and 180 mm, consequently the rise f of the vault was equal to:

$$f = 7.555 - 4.920 - (0.130 + 0.180)/2 = 2.48 \text{ m.}$$

The ratio of the rise and span was equal to $f/(2a) = {}^1/_8$ .The structure was composed of concrete fusées and reinforced with steel. In a section with a width of 1.0 m a number of eleven fusées were placed with a spacing of 10 mm. A fusée is a ceramic cylindrical element with a length of about 350 mm, an outward diameter of Ø80 mm and a thickness of 10 mm. The centre-to-centre distance of the ceramic elements is equal to 90 mm. The thickness of the vault was 130 mm. The vault was reinforced with bars Ø8 – 180 at the top and bottom. The Dutch building code GBV 1950 [C1] requires for floors with a thickness of 120 mm or more a cover on the reinforcement of at least 15 mm. To resist the thrust, steel bars Ø32 were made at a centre-to-centre distance of 1.0 m. To decrease the deformations the bars were supported with three ties Ø8 mm hanging down from the vault.

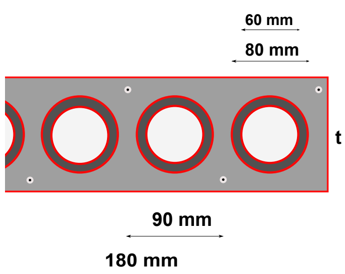

FIGURE 5.3 Section of the Fusée Céramique roof of building Q, thickness t =130 mm, reinforced with rebars Ø80-180.

Stiffness, area, second moment of the area and Young's modulus

For a part of the roof with width of 1.0 m the areas of the concrete, fusées and steel are calculated in table 5.1.

			Area
Fusées:	A_f =	$11 \times ¼\, \pi \times (80^2\text{-}60^2)$ =	24.190×10^3 mm²
Concrete:	A_c =	130×1000 - $11 \times ¼\, \pi\, 80^2$ =	74.708×10^3 mm²
Rebars at the top:	A_s =	$¼\, \pi\, 8^2 \times 1000/180$ =	279 mm²
Rebars at the bottom:	A_s =	$¼\, \pi\, 8^2 \times 1000/180$ =	279 mm²

TABLE 5.1 Area of the fusées, concrete and steel

For the concrete, steel and fusées the second moment of the area is calculated and shown in table 5.2. The thickness of the roof is equal to 130 mm, so the minimal cover is equal to 15 mm. Distribution bars were not used, the rebars Ø8 – 180 are positioned parallel to the span between the fusées with a covering of 15 mm at the top and bottom of the vault.

		Second moment of the area	[mm^4]
Concrete	I_c =	$1000 \times 130^3/12 - 11 \times \pi \times 80^4/64$ =	160.97×10^6
Fusées	I_f =	$11 \times \pi \times (80^4 - 60^4)/64$ =	15.12×10^6
Steel rebars 2×Ø8–180	I_s =	$2 \times 279 \times (½ \times 130 - 15 - ½ \times 8)^2$ =	1.18×10^6

TABLE 5.2 Second moment of area of the fusées, concrete and steel

According to the calculations made for a factory in Dongen, the Young's modulus of the fusées, reinforcement and concrete is equal to respectively $E_f = 1.7 \times 10^4$ MPa, $E_s = 2.1 \times 10^5$ MPa and $E_c = 2.1 \times 10^4$ MPa. The distribution and transfer of the loads is defined according to the Theory of Elasticity. The stiffness is calculated by multiplying the area and second moment of the area with the Young's modulus. To simplify the calculation a ratio n_f and n_s is introduced with: $n_f = E_f/E_s = 0.81$ and $n_s = E_s/E_c = 10$. Thus EA and EI are calculated with respectively:

$$EA = E_c\,(A_c + n_f A_f + n_s A_s) \qquad [5.1]$$

$$EI = E_c\,(I_c + n_f I_f + n_s I_s) \qquad [5.2]$$

Substituting the values for E_c, I_c, I_f and I_s into these equations gives for this vault:

$$EA = 2.1 \times 10^4 \times (74.708 \times 10^3 + 0.81 \times 24.19 \times 10^3 + 10 \times 2 \times 279) = 2.1 \times 10^9 \text{ N}$$

$$EI = 2.1 \times 10^4 \times (160.97 \times 10^6 + 0.81 \times 15.12 \times 10^6 + 10 \times 1.18 \times 10^6) = 3.89 \times 10^{12} \text{ Nmm}^2$$

Loads

According to the Dutch building code of 1955 for roofs the live load was assumed as $p_e = 0.5$ kN/m^2. The mass of the concrete and fusées is respectively 2400 kg/m^3 and 1800 kg/m^3. Thus the permanent load and live load is equal to:

Dead weight of the vault: $p_g = 74.704 \times 24 + 24.19 \times 18$ =	2.23	kN/m^2
Finishing:	0.11	kN/m^2
Tension bars:	0.06	kN/m^2
Permanent load:	2.40	kN/m^2
Live load:	0.50	kN/m^2

§ 5.2 Forces and bending moments

The Forces acting at the vault were calculated using the Theory of Elasticity. The vault was schematised as an arch resting on two simple supports. The stiffness of the supports was neglected; actually the ties will lengthen so the supports will move sideward. The thrust was calculated with the equilibrium of the bending moment around the top. Actually the bending moment at the top is not zero but the effect of this assumption is negligible as will be shown later.

Permanent load

The vault is subjected to the dead load q_g = 2.4 kN/m². The calculation is made for a width of 1.0 m. The span is equal to 19.8 m, half of the span is equal to a = 9.9 m. The vertical and horizontal reaction force acting on the supports are respectively:

$$V_A = V_B = q_g\, a = 2.4 \times 9.9 = 23.76 \text{ kN}; \quad H = \frac{q_g a^2}{2f} = \frac{2.4 \times 9.9^2}{2 \times 2.48} = 47.424 \text{ kN}$$

The vault is subjected to a compressive normal force; the normal force acting at the supports is equal to the sum of the vectors V and H:

$$N = (H^2 + V^2)^{0.5} = [47.424^2 + 23.76^2]^{0.5} = 53.04 \text{ kN}$$

For x = ½ a = 4.95 m the normal force follows from:

$$N = [H^2 + (q\,x)^2]^{0.5} = [47.424^2 + (2.4 \times 9.9/2)^2]^{0.5} = 48.9 \text{ kN}$$

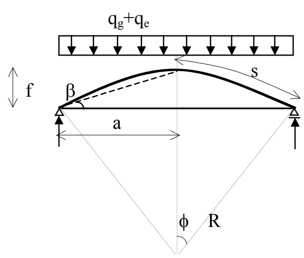

FIGURE 5.4 The vault subjected to a symmetrical load due to the permanent and live load.

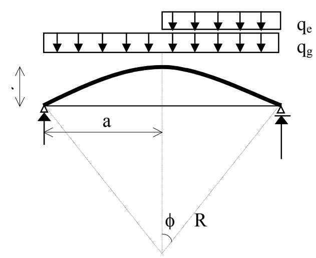

FIGURE 5.5 The vault subjected to the permant load and the asymmetric live load.

Full load

The vault is subjected to the dead load q_g = 2.4 kN/m² and a live load q_e = 0.5 kN/m². Due to the symmetrical live and dead load the vault is subjected to a load: q = 2.4 + 0.5 = 2.9 kN/m. The span is equal to *l* = 19.8 m, half of the span is equal to a = ½ *l* = 9.9 m. The vertical and horizontal reaction force acting on the supports are respectively:

$$V_A = V_B = (q_g + q_e)\, a = (2.4 + 0.5) \times 9.9 = 28.7 \text{ kN}; \quad H = \frac{(q_g + q_e)\, a^2}{2f} = \frac{(2.4 + 0.5) \times 9.9^2}{2 \times 2.48} = 57.3 \text{ kN}$$

The vault is subjected to a compressive normal force, the normal force acting at the supports is equal to the sum of the vectors V and H: $N = (H^2 + V^2)^{0.5} = [57.3^2 + 28.7^2]^{0.5} = 64.1$ kN

For x = ½ a = 4.95 m the normal force follows from:

$$N = [H^2 + (q\,x)^2]^{0.5} = [57.3^2 + (2.9 \times 9.9/2)^2]^{0.5} = 59.1 \text{ kN}$$

Asymmetric load

Due to an asymmetric load the vault is subjected to bending. Assume the vault is subjected to the live load acting at one side. The permanent load is equal to q_g = 2.4 kN/m². The live load is equal to q_e = 0.5 kN/m². The vault is subjected to a minimum load q_g = 2.4 kN/m² at one side and a maximum load equal to $q_g + q_e$ = 2.9 kN/m² at the other side. The expressions for an asymmetrical load acting at a parabolic vault are given in chapter 6.

The vertical and horizontal reaction force acting on the supports are respectively:

$V_A = q_g a + ¼ q_e a = 2.4 \times 9.9 + ¼ \times 0.5 \times 9.9 = 25.0$ kN

$V_B = q_g a + ¾ q_e a = 2.4 \times 9.9 + ¾ \times 0.5 \times 9.9 = 27.5$ kN

$$H = \frac{q_g a^2}{2f} + \frac{q_e a^2}{4f} = \frac{2.4 \times 9.92}{2 \times 2.48} + \frac{0.5 \times 9.92}{4 \times 2.48} = 52.4 \text{ kN}$$

The resulting normal forces acting at the supports are respectively:

$N_A = (H^2 + V^2)^{0.5} = (52.4^2 + 25.0^2)^{0.5} = 58.0$ kN

$N_B = (H^2 + V^2)^{0.5} = (52.4^2 + 27.5^2)^{0.5} = 59.2$ kN

The bending moment is equal to: $M_o = q_e a^2/16 = 0.5 \times 9.9^2/16 = 3.1$ kNm

For x = ½ a = 4.95 the normal force follows from:

$N = [H^2 + (V_B - q x)^2]^{0.5} = [52.4^2 + (27.5 - 2.9 \times 4.95)^2]^{0.5} = 54.0$ kN

Buckling

The rise of the parabolic vault is rather small, to define the buckling force the curve is approached by a segment of a circle. The radius of a circular vault arch is defined in chapter 9. For a span of *l* = 2 a = 19.8 m, a rise f = a/4 = 9.9/4 = 2.475 m the radius is equal to:

$$R = \frac{a^2 + f^2}{2f} = \frac{9.9^2 + 2.475^2}{2 \times 2.475} = 21.04 \text{ m}$$

The angle ϕ follows with a = 9.9 m and R = 21.04 m from :

$\sin \phi = a/R = 9.9/21.04 = 0.47 \rightarrow \phi = 0.49$ radians

Substituting this radius into the expression for the critical buckling force N_{cr} [2.6] gives:

$$N_{cr} = \frac{EI [\pi^2/\phi^2 - 1]}{R^2} = \frac{3.89 \times 10^{12} \times [\pi^2/0.49^2 - 1]}{(21.04 \times 10^3)^2} = 352 \times 10^3 \text{ N}$$

In reality the curvature is parabolic. For a shallow vault the form of a parabolic vault is very close to a circular arch and the difference is negligible. To demonstrate the accuracy of this approach, the buckling force is also calculated for a parabola. For a parabola the angle β between the tangent and the horizontal line through the supports can be calculated using:

$\tan 2\beta = 2 f/a = 2 \times 2.48/9.9 = 0.501 \rightarrow 2\beta = 26.61^0 = 0.465$ radians

The radius of a parabola varies. The radius for a parabola is defined with expression [2.9]: Substituting the rise f and a = 9.9 m into expression [2.9] gives:

$$R_\phi = \frac{a^2 (1 + 4 f^2/a^2)^{1/2}}{2f} = \frac{9.9^2 (1 + 4 \times 2.48^2/9.9^2)^{0.5}}{2 \times 2.48} = 22.1 \text{ m}$$

Substituting this radius into the expression for the critical buckling force N_{cr} [2.6] gives:

$$N_{cr} = \frac{EI [\pi^2/\phi^2 - 1]}{R_\phi^2} = \frac{3.89 \times 10^{12} \times [p^2/0.465^2 - 1]}{(22.1 \times 10^3)^2} = 355 \times 10^3 \text{ N}$$

For this shallow arch the buckling force for a circular arch is almost equal to the buckling force for a parabolic vault.

As described in chapter 3 Moon et al [Moo07] researched the critical buckling load for parabolic pin-ended arches. For in-plane asymmetric buckling mode the critical buckling load is for pin–ended arches ,with a length s from the top to the supports, equal to:

$$N_{cr\,asym} = \frac{\pi^2 EI}{s^2} \qquad [3.13]$$

The in-plane a-symmetric buckling load will be critical if the rise f meets the following condition:

$$f/a > 4.565\, i/a \qquad [3.14]$$

For: $EA = 2.1 \times 10^9$ Nmm² and $EI = 3.89 \times 10^{12}$ Nmm² the radius of gyration of the section is equal to:

$$i = (I/A)^{1/2} = (3.89 \times 10^{12} / 2.1 \times 10^9)^{1/2} = 43 \text{ mm}$$

For building Q the rise f is equal to 2.48 m. Snap through will be not critical if the rise fulfils the condition: $f > 4.565 \times 43 = 196$ mm, so for this structure the asymmetric buckling load will be decisive.

The length s of a parabolic vault from the crown to the supports is defined in chapter 2:

$$s = f\,(1+ \tfrac{1}{4}\, a^2/f^2)^{1/2} + \tfrac{1}{4}\, a^2/f \times \ln\{2\, f/a + (4\, f^2/a^2+1)^{1/2}\} \qquad [2.13]$$

Substitute $f = 2.48$ m, $a = 9.9$ m and $a/f = 4$:

$$s = 2.48 \times (1+ \tfrac{1}{4}\, .4^2)^{1/2} + \tfrac{1}{4} \times 9.9 \times 4 \times \ln\{2/4 + (4/16+1)^{1/2}\} = 5.545 + 4.763 = 10.31 \text{ m}$$

With [3.13] the critical buckling load is equal to:

$$N_{cr\,asym} = \frac{\pi^2 EI}{s^2} = \frac{\pi^2 \times 3.9 \times 10^{12}}{(10.31 \times 10^3)^2} = 361 \times 10^3 \text{ N}$$

For this shallow vault this result approximately equals the buckling load calculated using Timoshenko.

Buckling ratio, full load

For $x = \frac{1}{2} \times a$ the normal force is equal to $N = 59.1$ kN. Thus the ratio n_{cr} of the buckling force and the normal force is equal to: $n_{cr} = N_{cr}/N = 361/59.1 = 6.1$

Buckling ratio , asymmetrical load

For $x = \frac{1}{2} \times a$ the normal force is equal to $N = 54$ kN. Thus the ratio n_{cr} of the buckling force and the normal force is equal to: $n_{cr} = N_{cr}/N = 361/54 = 6.7$

Generally a ratio of $n_{cr} \geq 5$ was recommended. For the full load as well as the asymmetrical the ratio is larger than 5. So this structure meets the requirements. However time dependent effects and cracks will decrease the stiffness and critical buckling force over time. These effects will be considered later.

§ 5.3 Calculation of the normal stress.

Due to the normal force the concrete, fusées and steel are subjected to respectively a force N_c, N_f and N_s. The sum of these three forces is equal to the load: $N_c + N_f + N_s = N$.
According to the Theory of Linear Elasticity the normal force acting on the concrete, fusées or reinforcement follows from: $N_x = A E_x \varepsilon_x$. Substituting these forces in the expression for the equilibrium of the forces gives:

$$N = EA_c \varepsilon_c + EA_f \varepsilon_f + EA_s \varepsilon_s \qquad [5.3]$$

For a symmetrical loaded structure the specific deformation of the concrete, fusées and reinforcement will be equal, thus: $\varepsilon_c = \varepsilon_f = \varepsilon_s = \varepsilon_0$.

$$\varepsilon_0 = \frac{N_c}{EA_c} = \frac{N_f}{EA_f} = \frac{N_s}{EA_s} \qquad [5.4]$$

Substituting the specific deformation $\varepsilon_0 = N_c/EA_c$ in equation [5.3] gives:

$$N = EA_c \varepsilon_0 (1 + A_f n_f / A_c + A_s n_s / A_c)$$

$$\varepsilon_0 = \frac{N}{EA_c (1 + A_f n_f / A_c + A_s n_s / A_c)} = \frac{N}{EA_c m_{EA}} \qquad \text{with: } m_{EA} = (1 + A_f n_f / A_c + A_s n_s / A) \qquad [5.5]$$

According to the Theory of Elasticity the normal stress due to the normal force acting on the section of the vault is respectively for the concrete, fusées and reinforcement equal to: $\sigma_x = E_x \varepsilon_0$, Substituting the specific deformation into this expression gives:

$$\sigma_x = \frac{N E_x}{EA_c m_{EA}} \qquad [5.6]$$

Where: $E_c = 2.1 \times 10^4$ MPa; $E_f = 1.7 \times 10^4$ MPa; $E_s = 2.1 \times 10^5$ MPa; $n_f = E_f/E_c = 0.81$; $n_s = E_s/E_c = 10$;
$A_c = 74.708 \times 10^3$ mm^2; $A_f = 24.19 \times 10^3$ mm^2 ; $A_s = 2 \times 279$ mm^2.

$$m_{EA} = (1 + A_f n_f / A_c + A_s n_s / A_c) = 1 + \frac{24190 \times 0.81}{74708} + \frac{558 \times 10}{74708} = 1.34$$

$$EA = EA_c m_{EA} = 2.1 \times 10^4 \times 74.708 \times 10^3 \times 1.34 = 2.1 \times 10^9 \text{ N}$$

Permanent load:

Due to the permanent load the structure is subjected at a quarter of the span to a normal force equal to: N = 48.9 kN. The specific deformation is equal to:

$$\varepsilon_0 = \frac{N}{EA_c m_{EA}} = \frac{48.9 \times 10^3}{2.1 \times 10^9} = 2.33 \times 10^{-5}$$

The normal stresses and forces acting on respectively concrete, fusées and the reinforcement are:

concrete:	$\sigma_c = 0.0233 \times 10^{-3} \times 2.1 \times 10^4 =$	0.49 MPa	$N_c = 0.49 \times 74.708 \times 10^3 =$	36.6×10^3 N
fusées:	$\sigma_f = 0.0233 \times 10^{-3} \times 1.7 \times 10^4 =$	0.40 MPa	$N_f = 0.40 \times 24.19 \times 10^3 =$	9.6×10^3 N
rebars:	$\sigma_s = 0.0233 \times 10^{-3} \times 2.1 \times 10^5 =$	4.90 MPa	$N_s = 4.90 \times 558 =$	2.7×10^3 N

Normal and bending stress due to the asymmetrical load

Due to the dead load and the asymmetrical live load the structure is subjected to normal stresses and bending stresses. At a quarter of the span the normal force is for the permanent and asymmetrical live load equal to N = 54.0 kN. The specific deformation is equal to:

$$\varepsilon_0 = \frac{N}{EA_c\, m_{EA}} = \frac{54.0 \times 10^3}{2.1 \times 10^9} = 2.57 \times 10^{-5}$$

The stresses are calculated with the expression: $\sigma_x = \varepsilon_o E_x$ .The normal stresses and forces acting on respectively concrete, fusées and the reinforcement is equal to:

concrete:	$\sigma_c = 0.0257 \times 10^{-3} \times 2.1{\times}10^4 =$	0.54 MPa	$N_c = 0.54 \times 74.708{\times}10^3 =$	40.3×10^3 N
fusées:	$\sigma_f = 0.0257 \times 10^{-3} \times 1.7{\times}10^4 =$	0.44 MPa	$N_f = 0.40 \times 24.19{\times}10^3 =$	9.7×10^3 N
rebars:	$\sigma_s = 0.0257 \times 10^{-3} \times 2.1{\times}10^5 =$	5.40 MPa	$N_s = 5.40 \times 558 =$	3.0×10^3 N

Due to the asymmetrical loading the vault is subjected to a bending moment equal to:

$$M_o = \frac{q_e a^2}{16} = \frac{0.5 \times 9.9^2}{16} = 3.1 \text{ kNm}$$

Conforming to the Theory of Elasticity the stress is: $\sigma_c = \frac{M z E_c}{EI}$ [5.7]

Substituting EI = 3.89×10^{12} Nmm^2 and E_c = 21000 MPa into this expression results in:

$$\sigma_c = \frac{3.1 \times 10^6 \times 130/2 \times 21000}{3.89 \times 10^{12}} = 1.09 \text{ MPa}$$

The ratio of the buckling force to the normal force is equal to: $n_{cr} = N_{cr}/N = 361/54 = 6.7$

Due to second order effects the bending stresses are increased with a factor $n_{cr}/(n_{cr}-1)$:

$$\frac{n_{cr}}{n_{cr}-1}\sigma_c = \frac{6.7}{6.7-1} \times 1.09 = 1.28 \text{ MPa}$$

The maximum stresses due to compression and bending are thus equal to: $\sigma_c = -0.54 \pm 1.28$ MPa

The maximum compressive stress and maximum tensile stress are respectively: $\sigma_c = -1.82$ MPa and $\sigma_c = +0.74$ MPa . Due to the bending the sections are subjected to a bending tensile stress, probably sections halfway the top are cracked due to the asymmetrical live load.

§ 5.4 Calculation of the normal forces and bending moments with a computer program

To validate the analysis the normal forces and bending moments are calculated with a computer program, Matrix-frame. The structure is subjected to the permanent load, live loads, wind loads and snow loads. The equally distributed live load is equal to q = 0.5 kN/m.

Node	X	Z	Member	dy/dx	ds	Permanent load
1	-9.9	2.48				
2	-9.0	2.05	1-2	0.43	1.088	2.61
3	-8.0	1.62	2-3	0.38	1.07	2.57
4	-7.0	1.24	3-4	0.33	1.053	2.53
5	-6.0	0.91	4-5	0.28	1.038	2.49
6	-5.0	0.63	5-6	0.22	1.024	2.46
7	-4.0	0.41	6-7	0.18	1.016	2.44
8	-3.0	0.23	7-8	0.13	1.008	2.42
9	-2.0	0.1	8-9	0.07	1.002	2.41
10	-1.0	0.03	9-10	0.03	1.0	2.4
11	0	0	10-11	0	1.0	2.4
12	1.0	0.03	11-12	0	1.0	2.4
13	2.0	0.1	12-13	0.03	1.0	2.4
14	3.0	0.23	13-14	0.07	1.002	2.41
15	4.0	0.41	14-15	0.13	1.008	2.42
16	5.0	0.63	15-16	0.18	1.016	2.44
17	6.0	0.91	16-17	0.22	1.024	2.46
18	7.0	1.24	17-18	0.28	1.038	2.49
19	8.0	1.62	18-19	0.33	1.053	2.53
20	9.0	2.05	19-20	0.38	1.07	2.57
21	9.9	2.48	20-21	0.43	1.088	2.61

TABLE 5.3 Nodes, coordinates and permanent load for building Q

Permanent load

The permanent load acting at the top is equal to q = 2.4 kN/m. Due to the curvature the permanent load increases from the top to the supports with: q = 2.4 (ds/dx). The permanent load acting at the supports is:

$$q_{x=a} = 2.4\,[1 + (2\,f/a)^2]^{1/2} = 2.61 \text{ kN/m.}$$

Wind loads

The wind load acting on the roof is calculated according to the Euro code NEN EN 1991-1-4-2005. The city of Woerden is situated at the frontier between South Holland and Utrecht. The height of the structure above ground level is 7.55 m. For an urban area II, partly without adjacent buildings, the wind load is for z = 7.0 m and for z = 8.0 m respectively: for z = 7.0 m: q(z) = 0.75 kN/m^2, for z = 8.0 m: q(z) = 0.79 kN/m^2. Interpolation gives for z = 7.55: q(z) = 0.75 + 0.04 × 0.55 = 0.77 kN/m^2. The coefficients for internal pressure are for overpressure c = + 0.2 and for under pressure c = - 0.3.
For the coefficients to define the external wind load three areas are distinguished: A windward side, B at the top and C leeward side:

A windward side:	c= -1.2 and c = + 0.1
B at the top, sucking:	c = -0.82
C leeward side, sucking:	c = -0.4

For sucking windloads the sign is negative, for compressive windload the sign is positive.

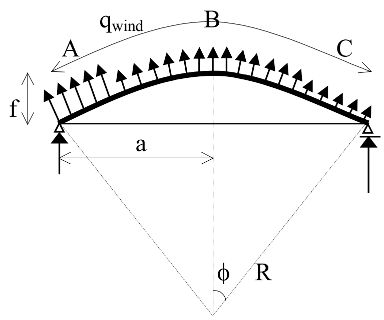

FIGURE 5.6 Wind load acting on the vault, zone A at the windward side, zone B at the top and zone C at the leeward side

Combining internal and external wind pressure two extreme wind loads arise, respectively with over pressure and under pressure:

	over pressure	under pressure
A	$p_w = (-1.2 - 0.2) \times 0.77 = -1.08$	$p_w = (+0.1 + 0.3) \times 0.77 = +0.31$
B	$p_w = (-0.82 - 0.2) \times 0.77 = -0.79$	$p_w = (-0.82 + 0.3) \times 0.77 = -0.40$
C	$p_w = (-0.4 - 0.2) \times 0.77 = -0.46$	$p_w = (-0.4 + 0.3) \times 0.77 = -0.08$

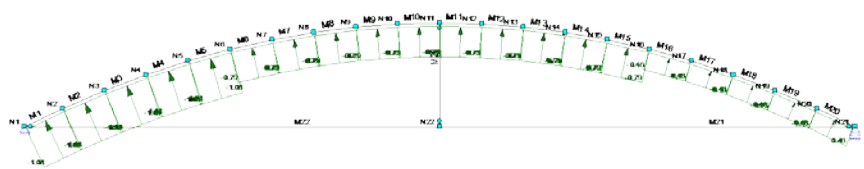

FIGURE 5.7 Wind load with over pressure

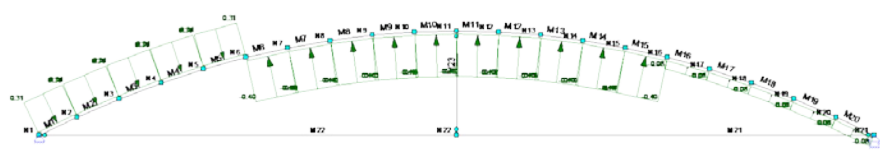

FIGURE 5.8 Wind load with under pressure

Snow loads

According to the NEN-EN 1991-1-3 the roof is subjected to a snow-load: $p_{sn} = u_3 \times 0.7$ kN/m².
The coefficient u_3 depends on the ratio of the rise versus the span. Two alternatives are distinguished, the snow load can be equally distributed or linearly increasing, see figure 5.9.

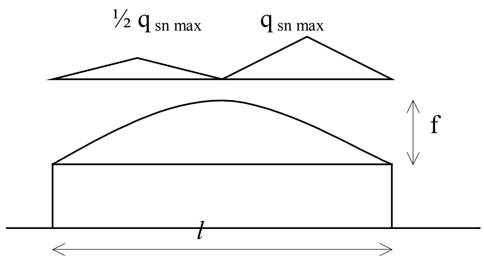

FIGURE 5.9 Linear increasing snow load with maximum at ¼ *l*

For an equally distributed load and a cylindrical roof with a ratio $f/l = {}^1/_8$ the coefficient u_3 is equal to $u_3 = 0.8$: $p_{sn} = 0.8 \times 0.7 = 0.56$ kN/m².

For a linearly increasing snow load, with a maximum at ½ a, u_3 follows from: $u_3 = 0,2 + 10 \times f/l$

For $f/l = 0,125$ the coefficient u_3 is equal to: $u_3 = 0,2 + 10 \times 0.125 = 1.45$

The maximum load acting at the roof is: $p_{sn\,max} = 1.45 \times 0.7 = 1.02$ kN/m².

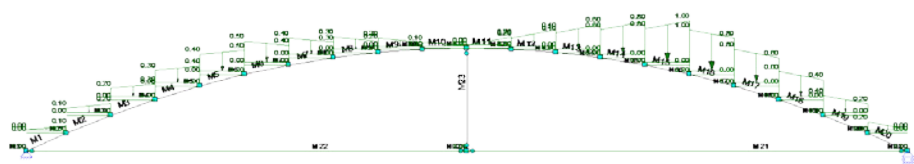

FIGURE 5.10 Linear increasing snow load with maximum at x = a/2.

Output

The following figures and table show the results of the finite element analysis with Matrix-frame. Due to the permanent load the bending moments are pretty small. The maximum bending moment, M = 3.,1 kNm, is found for the variable load acting at one side of the vault. The bending moments due to the snow and wind loads are slightly smaller than the bending moments due to the live load. The results are very close to those calculated earlier. The bending moment calculated using the finite element technique for the live load is equal to 3.09 kNm, this moment is identically to the bending moment calculated with the analytical method. The maximum bending moment due to the surface load is 0.51 kNm. This bending moment is much less than the bending moment resulting from the live load. Due to the wind load the tie is compressed, fortunately the compressive force is compensated by the tensile force due to the permanent load.

For the preliminary design of a vault in an early stage of the process it is often labour efficient to reduce the number of possible load combinations and design the structure with only one or two critical load combinations. For the design of this roof, subjected to rain, snow or wind , an asymmetric variable load of $p_e = 0.5$ kN/m² will be the critical variable load.

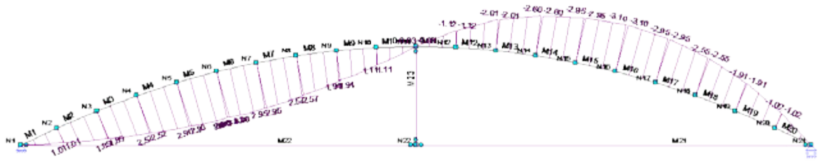

FIGURE 5.11 Bending moments due to the variable load q = 0.5 kN/m.

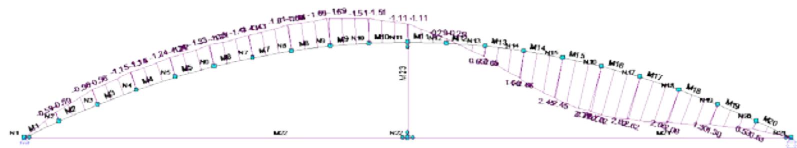

FIGURE 5.12 Bending moments due to the linearly increasing snow load.

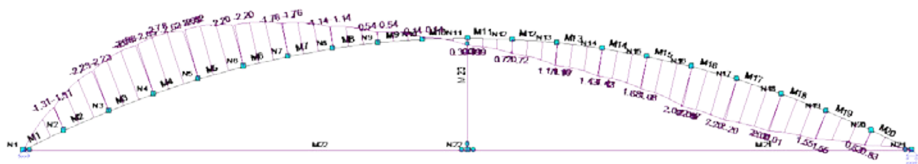

FIGURE 5.13 Bending moments due to the wind load over pressure

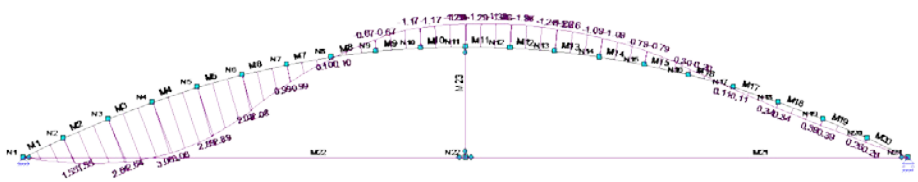

FIGURE 5.14 Bending moments due to the wind load under pressure

Member	Load:	N_x	V_z	M
4	Permanent load	- 51.18	1.33	0.45
	Live load	- 5.41	0.60	2.90
	snow load	- 8.29	0.26	1.23
	wind over pressure	16.48	0.59	2.65
	wind under pressure	5.93	0.34	3.08
10	Permanent load	- 48.33	1.45	0.51
	Live load	- 4.93	1.39	1.11
	snow load	- 7.87	0.43	1.50
	wind over pressure	16.61	0.93	0.39
	wind under pressure	5.72	0.33	1.30
16	Permanent load	- 50.42	1.40	0.26
	Live load	- 5.10	0.14	3.09
	snow load	- 8.31	0.56	2.82
	wind over pressure	16.72	0.34	2.19
	wind under pressure	5.78	0.44	0.30
22, tie	Permanent load	48.29		
	Live load	4.95		
	snow load	7.85		
	wind over pressure	-15.29		
	wind under pressure	- 5.32		

TABLE 5.4 Results, normal forces, shear forces and bending moments

§ 5.5 Calculation of the stresses including time dependent effects

The calculations shown are based on the Theory of Elasticity. Due to shrinkage and creep a structure of concrete will deform considerably. For a structure composed of several materials with varying features these time dependent effects will change the distribution of the loads. Especially the stresses due to the permanent loading are effected by the time dependent effects caused by shrinkage and creep. For most roofs the live loads act only for a short time on the structure, so for these loads the creep can be neglected. The dead load is acting on the structure during the life time, so the stresses due to the dead load are affected by the time dependent effects. For the parabolic vault, the stresses are calculated for the permanent loads acting directly as well as acting over a longer period, including time dependent effects.

For the vault with a width of 1.0 m Young's modulus, the Area and the second moment of the Area are respectively for the concrete, fusées and rebars:

Concrete	E_c = 21000 MPa	$A_c = 74.708 \times 10^3$ mm²	$I_c = 160.97 \times 10^6$ mm⁴
Fusées	E_f = 17000 MPa	$A_f = 24.190 \times 10^3$ mm²	$I_f = 15.12 \times 10^6$ mm⁴
Steel	$E_s = 2.1 \times 10^5$ MPa	A_s = 558 mm²	$I_s = 1.18 \times 10^6$ mm⁴

Immediately deformation, t = 0, permanent load

For a width of 1.0 m the permanent load is equal to: q_g = 2.4 kN/m.
For x = 4.95 m the resulting normal force is equal to N = 48.9 kN.
Due to this normal force the specific deformation is: $\varepsilon_0 = \dfrac{48900}{2.1 \times 10^9} = 0.0233 \times 10^{-3}$

The normal stresses and forces acting on respectively concrete, fusées and the reinforcement is equal to:

concrete:	$\sigma_c = 0.0233 \times 10^{-3} \times 2.1\times10^4 =$	0.49 MPa	$N_c = 0.49 \times 74.708\times10^3 =$	36.6×10^3 N
fusées:	$\sigma_f = 0.0233 \times 10^{-3} \times 1.7\times10^4 =$	0.40 MPa	$N_f = 0.40 \times 24.19\times10^3 =$	9.6×10^3 N
rebars:	$\sigma_s = 0.0233 \times 10^{-3} \times 2.1\times10^5 =$	4.90 MPA	$N_s = 4.90 \times 558 =$	2.7×10^3 N

Time dependent deformation

Creep and shrinkage will increase the deformation of the structure with $\Delta\varepsilon$. For a time t = ∞ the specific deformation of the structure is equal to: $\varepsilon_{t=\infty} = \varepsilon_0 + \Delta\varepsilon$. Due to shrinkage the concrete and the fusées shorten. The specific deformation of the concrete due to the shrinkage is named ε_{rc}. The steel and fusées don't shrink. Due to the shrinkage of the concrete the joints of the fusées will deform. The assumption is made that the specific deformation of the fusées due to the deformation of the joints caused by the shrinkage of the concrete is equal to $\varepsilon_{rf} = 0.1 \times 10^{-3}$. A decrease of this deformation will increase the internal forces.

Shrinkage

According to the NEN-EN 1992-1-1 Eurocode 2 the specific deformation of the concrete due to shrinkage is calculated with:

$$\varepsilon_{rc} = \beta(t,t_0)\ k_h\, \varepsilon_{rc} \quad \text{and} \quad \beta(t,t_0) = \frac{t - t_0}{t - t_0 + 0.04 \times \sqrt{h_0}}$$

A section of the vault has a thickness of 130 mm and a width of about 5.0 m, the specific thickness follows from h_0 = 2 A/u, with u is the circumference.

$$h_0 = \frac{2 \times 130 \times 5000}{2 \times (5000 + 130)} = 127 \text{ mm}$$

According to NEN-EN 1992-1-1 Eurocode 2 table 3.3: for h_0 = 100 mm, k_h = 1.0
for h_0 = 200 mm, k_h = 0.85

Interpolation for h_0 = 127 mm gives: k_h = 0.85 + 0.5 × (200-127)/100 = 0.96

The shrinkage is affected by the relative humidity, according to NEN-EN 1992-1-1 Eurocode 2 table 3.2. the shrinkage is:
For RH = 60% the specific shrinkage is: $\varepsilon_{rc} = 0.49 \times 10^{-3}$
For RH = 80% the specific shrinkage is: $\varepsilon_{rc} = 0.30 \times 10^{-3}$

The warehouse was slightly heated, the RH was approximately 70%, then the specific shrinkage would be:

$$\varepsilon_{rc} = \beta(t,t_0)\, k_h\, \varepsilon_{rc} \quad \text{with: } \beta(t,t_0) = \frac{t - t_0}{t - t_o + 0.04\sqrt{h_o}}$$

$$\text{for } t = \infty\text{: } \beta(t,t_0) = 1 \qquad \varepsilon_{rc} = 1 \times 0.96 \times ½ \times (0.49 + 0.3)\, 10^{-3} = 0.38 \times 10^{-3}$$

Creep

Due to creep a concrete structure subjected to a compressive load will shorten during its life time. Depending on the environment the creep of concrete is about $\phi = 1$ to 4. According to the NEN-EN 1992-1-1 the specific creep is calculated. The moulds were reused as fast as possible, but the time before the mould could be removed had to be at least 36 hours and the strength of the concrete had to be developed enough at this stage. To allow for this probably cement class N was used and the time t_0 was at least 2 days. The warehouse was slightly heated, the RH was approximately 70%.

According to NEN-EN 1992-1-1 figure 3.1: for RH = 50% the specific creep is $\phi(\infty,t_0) = 4.8$
for RH = 80% the specific creep is $\phi(\infty,t_0) = 3.6$

For RH = 70% the specific creep is equal to:

$$\phi(\infty,t_0) = 3.6 + \frac{(80 - 70) \times (4.8 - 3.6)}{(80 - 50)} = 4.0$$

For a concrete structure subjected to a permanent compressive load the instantaneous specific deformation is ε_o. Due to creep the specific deformation will increase with $\phi\, \varepsilon_o$. The total deformation of the concrete is equal to $\varepsilon_o(1+\phi)$. For $\phi(\infty,t_0) = 4$ the total deformation is: $\varepsilon_o(1+4)$.
Due to a normal load N acting on the structure composed of fusées, steel and concrete, the fusées, steel and concrete will be compressed. Due to creep and shrinkage the deformation of the concrete part increases. The concrete, fusées and rebars are tightly connected, the deformation of the concrete must necessarily be equal to the deformation of the fusées and steel, so the load is redistributed. The redistribution is calculated as following. For a structure composed of concrete, steel and fusées the deformations of the three materials are equal, so due to the shortening of the concrete the fusées and rebars must also shorten. The steel and fusées are subjected to internal compressive forces respectively F_s and F_f. The concrete is subjected to an internal tensile force F_c. The internal forces are in balance, thus: $F_c = F_f + F_s$

The magnitude of the internal forces follows from the equations describing the equilibrium of forces and the compatibility.

The specific deformation for the concrete is equal to:

$$\varepsilon_{t=\infty} = \varepsilon_0 + \Delta\varepsilon = \varepsilon_0 + \varepsilon_0\,\phi + \varepsilon_{rc} - \frac{F_c\,(1 + k\,\phi)}{AE_c}$$

The specific deformation for the fusées is equal to: $\varepsilon_{t=\infty} = \varepsilon_0 + \Delta\varepsilon = \varepsilon_0 + \varepsilon_{rf} + \frac{F_f}{AE_f}$

The specific deformation for the rebars is equal to: $\varepsilon_{t=\infty} = \varepsilon_0 + \Delta\varepsilon = \varepsilon_0 + \frac{F_s}{AE_s}$

With these expressions an expression to calculate the force F_s is defined, see chapter 4.

$$F_s = \frac{AE_s\,[\,(\varepsilon_0\,\phi + \varepsilon_{rc})\,A_c/(1+k\,\phi) + \varepsilon_{rf}A_f\,n_f\,]}{A_c/(1+k\,\phi) + A_f\,n_f + A_s\,n_s}$$

$$\frac{F_s}{A_sE_s} = \frac{(0.0233\times10^{-3}\times 4 + 0.38\times10^{-3})\times74708/(1+½\times 4)+0.1\times10^{-3}\times24190\ 0.81}{74.708\times10^3/(1+½\times 4) + 24.19\times10^3\times 0.81 + 558\times 10} = 0.275\times10^{-3}$$

$$F_s = 0.275\times10^{-3}\times 558\times 2.1\times10^5 = 32.2\times10^{-3}\ \text{N}$$

With F_s the force F_f acting in the fusées follows from [4.23]: $F_f = \frac{F_s\,AE_f}{AE_s} - \varepsilon_{rf}\,AE_f$

$$F_f = \frac{32200\times 24.19\times10^3\times 17000}{558\times 2.1\times 10^5} - 0.1\times10^{-3}\times 24.19\times10^3\times 17000 = 71.9\times10^3\ \text{N}$$

The force F_c follows from: $F_c = F_f + F_s$ → $F_c = 71.9 + 32.2 = 104.1$ kN

After a long time, t = ∞, the forces acting on the structure composed of concrete, fusées and steel change and become respectively:

The force acting on the concrete is equal to: $N_c - F_c = -36.6 + 104.1 = +67.5\times10^3$ N
The force acting on the fusées concrete is equal to: $N_f + F_f = -9.6 - 71.9 = -81.5\times10^3$ N
The force acting on the reinforcement is equal to: $N_s + F_s = -2.7 - 32.2 = -34.9\times10^3$ N

Probably the tensile forces acting on the concrete are increased due to varying thermal expansion of the fusées and concrete.

Cracked structure

The concrete is tensioned, so it is likely that the concrete will crack due to the tensile stress. In a crack the tensile force acting in the concrete section is zero, so: $N_c - F_c = 0$ In a crack the load is transferred by the fusées and the reinforcement: $N = (N_c - F_c) + (N_f + F_f) + (N_s + F_s)$

Substituting $N_c - F_c = 0$ and $N = N_c + N_f + N_s$ into this expression gives: $N_c = F_f + F_s$ [5.8]

The specific deformation of the fusées is equal to the specific deformation of the steel:

$$\varepsilon_t = \varepsilon_0 + \frac{F_s}{AE_s} = \varepsilon_0 + \varepsilon_{rf} + \frac{F_f}{AE_f} \quad\rightarrow\quad F_f = AE_f\left(\frac{F_s}{AE_s} - \varepsilon_{rf}\right)$$

Substituting F_f into [5.8] to calculate F_s:

$$N_c = AE_f\left(\frac{F_s}{AE_s} - \varepsilon_{rf}\right) + F_s \quad\rightarrow\quad F_s = \frac{AE_f\varepsilon_{rf} + N_c}{AE_f/(AE_s) + 1}$$

With $N_c = 36.6\times10^3$ N the force F_s is:

$$F_s = \frac{24.19\times10^3\times1.7\times10^4\times0.1\times10^{-3} + 36.6\times10^3}{24.19\times10^3\times1.7\times10^4/(558\times2.1\times10^5) + 1} = 17.2\times10^3\ \text{N}$$

The force acting at the fusées follows from: $F_f = AE_f\left(\frac{F_s}{AE_s} - \varepsilon_{rf}\right)$ →

$$F_f = 24.19 \times 10^3 \times 1.7 \times 10^4 \times \left(\frac{17.2 \times 10^3}{558 \times 2.1 \times 10^5} - 0.1 \times 10^{-3} \right) = 19.2 \times 10^3 \text{ N}$$

The force acting on the concrete is equal to: $N_c = 36.6 - 36.6 = 0$
The force acting on the fusées is equal to: $N_f + F_f = -9.6 - 19.2 = -28.8$ kN
The force acting on the reinforcement is equal to: $N_s + F_s = -2.7 - 17.2 = -19.9$ kN

In a cracked section the stresses acting in the fusées and rebars are decreased. The time dependent values are calculated for a creep factor $\phi = 4$. If the concrete is cracked then the stress is practically zero, so the concrete does not transfer any load. The concrete is only shortened by creep in case it is subjected to a load, so the creep only has an effect before cracking. Thus the effect of the creep is very limited. The resulting forces and stresses (excluding second order effects) are given in the following tables. The stresses acting into the cracked section are less than in the uncracked section.

	Normal force [kN]	internal force [kN]	resulting force[kN]	uncracked section stress [MPa]	cracked section force [kN]	cracked section stress [MPa]
Concrete:	-36.6	+ 104.1	+ 67.5	+ 0.9	0	0
Fusées:	-9.6	– 71.9	- 81.5	- 3.4	-28.8	- 1.2
Rebars:	-2.7	- 32.2	- 34.9	- 63.0	-19.9	-35.7

TABLE 5.5 Resulting forces and stresses due to the permanent load, including time dependant effects $t = \infty$, for a uncracked section and cracked section.

§ 5.6 Stiffness

The stiffness of a structure of concrete can be calculated with a MN-κ diagram. The curvature κ of a structure loaded by a bending moment and a normal force is defined with:

$$\kappa = \frac{\varepsilon_c}{x} = \frac{\sigma_c}{E_c k_x h} \quad [5.9]$$

With:
h = the height of the section
x = the height of the compresive zone
$k_x = x/h$

Next the stiffness is calculated with: $EI = M/\kappa$ [5.10]

The stiffness is minimal if the bending moment is at maximum. For a structure subjected to a relative small normal force the bending moment is at maximum if the stress in the reinforcement in the tensioned zone is at maximum, so $\sigma_s = f_s$. Then the curvature follows from:

$$\kappa_{et} = \frac{f_s/E_s}{(1 - d/h - k_x)\, h} \quad [5.9']$$

The section of a Fusée Céramique vault is not massive. To include the effect of the infill an imaginary force F_f acting at the interior of the fusées is defined. The stresses, forces and specific deformations in the concrete, reinforcement and infill follow from respectively:

Concrete, compressive zone:	$\sigma_c = E_c \varepsilon_c$	$F_c = ½ k_x b h \sigma_c$	ε_c
Reinforcement at the compressed side:	$\sigma_{sc} = E_s \varepsilon_{sc}$	$F_{sc} = ½ \omega b h \sigma_{sc}$	$\varepsilon_{sc} = \varepsilon_c (k_x - d/h)/k_x$
Reinforcement at the tensioned side:	$\sigma_{st} = E_s \varepsilon_{st}$	$F_{st} = ½ \omega b h \sigma_{st}$	$\varepsilon_{st} = \varepsilon_c (1 - k_x - d/h)/k_x$
Infill, for $k_x > c_f$	$\sigma_f = E_c \varepsilon_f$	$F_f = b m r^2 \sigma_f$	$\varepsilon_f = \varepsilon_c (k_x - c_f)/k_x$

For a section subjected to a force N the equilibrium of forces and bending moment is respectively:

$$N = F_c - F_f + F_{sc} - F_{st} \quad [5.11]$$

$$M = F_c (½ h - k_x/3) + (F_{sc} + F_{st}) \times (½ h - d) - F_f (r - z) \quad [5.12]$$

For a structure subjected to a given normal force the stiffness is defined with a MN-κ diagram. This diagram is made with the following procedure for a compressive zone k_x decreasing from a maximum value to a minimal value. For the concrete ,to include the effect of the time dependent deformations and varying thermal expansion of the fusées and concrete, the tensile stresses are neglected.
Firstly the stresses and forces are defined with the equilibrium of forces [5.11]. Next the bending moment is calculated [5.12]. The curvature follows from expression [5.9]. The stiffness is defined with the MN-κ diagram for a given bending moment with expression [5.10]. Successively for a decreasing value of the compressive zone the stresses, forces and bending moments are defined.

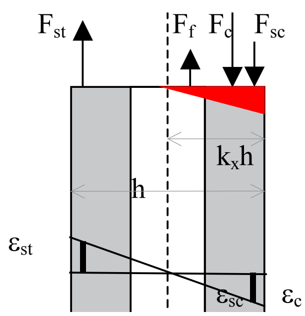

FIGURE 5.15 Forces and deformations in a section of the fusee vault for $k_x = ½$

Maximum compressive zone

The compressive zone is at maximum: $k_x = \infty$. The bending moment is at minimum.
The forces acting at the concrete is equal to: $F_c = b h \sigma_c$
The imaginary force acting at the interior of the fusées is: $F_c = \pi m r^2 \sigma_c$
where: m is the number of fusées
r is the radius of the interior of the fusées.
The force acting on the reinforcement in the compressed and tensioned zone is respectively:

$$F_{sc} = F_{st} = ½ n_s \omega b h \sigma_c$$

The compressive stress acting in the concrete follows from the equilibrium of forces [5.11]

$$N = \sigma_c (b h - \pi m r^2 + n_s \omega b h) \quad \rightarrow \quad \sigma_c = \frac{N}{(b h - \pi m r^2 + n_s \omega b h)}$$

The bending moment and curvature are zero: M = 0 and κ = 0.

The compressive zone is equal to the height of the section

Assume the compresive zone is equal to the height of the section: $x = h$ and $k_x = 1$. The stresses and forces acting at the concrete, infill and compressed rebars are respectively equal to:

The force acting on the concrete is: $F_c = ½\, k_x\, b\, h\, \sigma_c$

The imaginary force acting at the infill is: $F_f = -½\, m\, \pi\, r^2\, \sigma_c$

The force acting on the rebars at the compressed side: $F_{sc} = ½\, n_s\, \omega\, b\, h\, (1 - d/h)\, \sigma_c$

The force acting on the rebars at the other side: $F_{st} = ½\, n_s\, \omega\, b\, h\, (d/h)\, \sigma_c$

The compressive stress acting in the concrete follows from the equilibrium of forces [5.11]

$$N = ½\, k_x\, b\, h\, \sigma_c - ½\, m\, \pi\, r^2\, \sigma_c + ½\, n_s\, \omega\, b\, h\, \sigma_c$$

The maximum stress acting at the concrete is equal to:

$$\sigma_c = \frac{N}{½\, k_x\, b\, h - ½\, m\, \pi\, r^2 + ½\, n_s\, \omega\, b\, h}$$

The bending moment follows from [5.12]:

$$M = F_c\, (½\, h - {}^1/_3\, h) - 2 \times {}^2/_3\, m\, r^2 \sigma_c\, (r/h) \times 0.589\, r + ½\, n_s\, \omega\, b\, h\, (1 - 2\, d/h)\, \sigma_c\, (½\, h - d)$$

Next the curvature and stiffness are calculated with [5.9] and [5.10].

The stress in the tensioned rebars is zero

Assume the stress in the rebars at the tensioned zone is zero. The width of the compressive zone is equal to $x = k_x\, h = h - d$. The stresses and forces acting at the concrete, infill and compressed rebars are respectively equal to:

The force acting on the concrete is: $F_c = ½\, k_x\, b\, h\, \sigma_c$

The imaginary force acting at the infill is: $F_f = -\dfrac{m\, \pi\, r^2\, (k_x - ½)\, \sigma_c}{k_x}$

The force acting on the rebars at the compressed side: $F_{sc} = ½\, n_s\, \omega\, b\, h\, (k_x - d/h)\, \sigma_c/k_x$

The compressive stress acting in the concrete follows from the equilibrium of forces [5.11]

$$N = ½\, k_x\, b\, h\, \sigma_c - m\, \pi\, r^2\, (k_x - ½)\, \sigma_c/k_x + ½\, n_s\, \omega\, b\, h\, (k_x - d/h)\, \sigma_c/k_x$$

The maximum stress acting at the concrete follows from:

$$\sigma_c = \frac{N}{½\, k_x\, b\, h - m\, \pi\, r^2\, (k_x - ½)/k_x + ½\, n_s\, \omega\, b\, h\, (k_x - d/h)/k_x}$$

The bending moment follows from [5.12]:

$$M = F_c\, h\, (½ - k_x/3) - 2 \times [{}^2/_3\, m\, r^2\, \sigma_c\, r/(k_x\, h)] \times 0.589\, r + F_{sc}\, (½\, h - d)$$

Next the curvature and stiffness are calculated with [5.9] and [5.10].

Compressive zone equal to the width of the flange and the diameter of the fusées

The stiffness is calculated for a compressive zone $x = k_x h$ with $k_x = 1 - c_f/h$, assuming $\sigma_c < f_c$, $\sigma_{sc} < f_s$ and $\sigma_{st} < f_s$.

The force acting on the concrete is equal to: $F_c = ½ k_x b h \sigma_c$

The force acting at infill is: $F_f = \dfrac{- m \pi r^2 \sigma_c (k_x - ½)}{b h k_x}$

Rebars at the compressed side: $F_{sc} = ½ n_s \omega (k_x - d/h) \sigma_c / k_x$

Rebars at the tensioned side: $F_{st} = ½ n_s \omega (1 - k_x - d/h) \sigma_{c/kx}$

The compressive stress acting in the concrete follows from the equilibrium of forces [5.11]

$$\frac{N}{b h} = ½ k_x \sigma_c - \frac{m \pi r^2 (k_x - ½) \sigma_c}{b h k_x} + \frac{½ n_s}{} \frac{\omega (2 k_x - 1) \sigma_c}{k_x}$$

The maximum stress acting at the concrete follows from:

$$\sigma_c = \frac{N}{½ k_x b h - m \pi r^2 (k_x - ½)/k_x + ½ n_s \omega b h (2 k_x - 1)/k_x}$$

Provided the stress acting in the concrete and reinforcement does not exceed the maximum stress (thus $\sigma_c < f_c$, $\sigma_{sc} < f_s$ and $\sigma_{st} < f_s$) the bending moment follows from:

$$M = F_c (½ h - k_x/3) + (F_{st} + F_{sc}) \times (½ h - d) - \frac{2 \times {}^2/_3 m r^3 \sigma_c \times 0.589 r}{k_x h}$$

The curvature and stiffness is calculated respectively with [5.9] and [5.10].

Compressive zone is equal to half of the height of the section

Assume the compressive zone is equal to half of the height of the section: $x = k_x h = ½ h$.

The force acting in the concrete is equal to: $F_c = ¼ b h \sigma_c$

The force acting at the infill is: $F_f = - \dfrac{{}^2/_3 m r^3 \sigma_c}{½ b h^2}$

Rebars at the compressed side: $F_{sc} = \dfrac{½ n_s \omega (½ - d/h) \sigma_c}{k_x}$

Rebars at the tensioned side: $F_{st} = \dfrac{½ n_s \omega (½ - d/h) \sigma_c}{k_x}$

The compressive stress acting in the concrete follows from the equilibrium of forces [5.11]

$$\frac{N}{b h} = ¼ \sigma_c - \frac{{}^4/_3 m r^3 \sigma_c}{b h^2} \quad \rightarrow \quad \sigma_c = \frac{N}{¼ b h - {}^4/_3 m \pi r^3/h}.$$

Provided the stress acting in the concrete and reinforcement does not exceed the maximum stress (thus $\sigma_c < f_c$, $\sigma_{sc} < f_s$ and $s_{st} < f_s$) the bending moment follows from:

$$M = F_c (½ h - k_x/3) + (F_{st} + F_{sc}) \times (½ h - d) - ({}^4/_3 m r^3 \sigma_c/h) \times 0.589 r$$

The curvature and stiffness is calculated with respectively [5.9] and [5.10].

Compressive zone smaller than half of the height

Assume the compressive zone is smaller than half of the height of the section: $k_x = ½$ and the steel stress is smaller than the maximum stress: $\sigma_{st} < f_s$.

The force in the reinforcement in the tensioned zone is: $F_{st} = ½ \omega b h \sigma_s$

The concrete force is: $F_c = ½ b k_x h \sigma_c$

The stress in the infill is: $F_f = - m \beta r^2 \sigma_f$

For $k_x \leq ½$: $\sigma_f = \dfrac{\sigma_c (k_x - c_f/h)}{k_x}$ and $\beta = \dfrac{(\sin\phi - \phi\cos\phi - \sin^3\phi/3)}{1 - \cos\phi}$

The angle ϕ describes the sectional surface of the tubular infill, with $0 < \phi < \pi$, subjected to a linearly stress.

The stress in the reinforcement in the compressive zone is: $\sigma_{sc} = n_s \sigma_c (k_x - d/h)/k_x$

The stress in the reinforcement in the tensioned zone is: $\sigma_{st} = n_s \sigma_c (1 - k_x - d/h)/k_x \leq f_s$

The equilibrium of the vertical forces is:

$$\frac{N}{b h} = ½ \sigma_c k_x - \frac{\sigma_c (k_x - c_f/h) m \beta r^2}{k_x b h} + \frac{\sigma_c ½ n_s \omega (2 k_x - 1)}{k_x}$$

The compressive stress acting in the concrete follows from the equilibrium of forces:

$$\sigma_c = \frac{N}{½ k_x b h - (k_x - c_f/h) m \beta r^2/k_x + ½ n_s b h \omega (2 k_x - 1)/k_x}.$$

The bending moment follows from:

$$\frac{M_e}{bh^2} = ½ k_x \sigma_c [(½ - k_x/3) - \frac{\sigma_c (k_x - c_f/h) m \beta r^2 (r - z)}{k_x b h^2} + ½ \omega (\sigma_{st} + \sigma_{sc}) (½ - d/h)$$

with: $$z = \frac{{}^2/_3 \sin^3\phi (1 + ¼\cos\phi) - (¼ + \cos\phi) \times (\phi - \sin\phi\cos\phi)}{\sin\phi - \phi\cos\phi - \sin^3\phi/3}$$

For $k_x = (½ r + c_f)/h$: $\phi = \pi/3$, $\beta = 0.252$; $r - z = 0.79 r$

For $k_x = ½$: $\beta = {}^2/_3$; $r - z = 0.589 r$

The curvature and stiffness is calculated respectively with [5.9] and [5.10].

Steel stress exceeds the maximum stress

If the steel stress execeeds the maximum stress then the forces and bending moments are calculated, assuming the steel stress in the tensioned zone is at maximum, $\sigma_s = f_s$. The compressive stress acting in the concrete follows from the equilibrium of forces:

$$\sigma_c = \frac{N/(b h) + ½ \omega b h f_s}{½ k_x b h - (k_x - c_f/h) m \beta r^2/k_x + ½ n_s b h \omega (k_x - d/h)/k_x}$$

The moment follows from:

$$\frac{M_e}{b h^2} = ½ k_x \sigma_c (½ - k_x/3) - \frac{\sigma_c (k_x - c_f/h) m \beta r^2 (r - z)}{k_x b h^2} + ½ \omega (f_s + \sigma_{sc}) (½ - d/h)$$

Next the curvature is calculated with [5.11]: $$\kappa_{et} = \frac{f_s/E_s}{(1 - d/h - k_x) h}.$$

Finally the stiffness is found with [5.10]: $EI = M_e/\kappa_e$

Compressive zone smaller than the width of the flange

The force acting on the infill is zero for a compressive zone smaller than the width of the flange, $k_x < c_f/h$. Assume the concrete stress is smaller than the maximum stress and the stress acting in the reinforcement is smaller than the maximum stress, thus $\sigma_c < f_{c'}$, $\sigma_{sc} < f_s$, and $\sigma_{st} < f_s$.

The stress in the reinforcement in the compressive zone is: $\sigma_{sc} = n_s \sigma_c (k_x - d/h)/k_x$

The stress in the reinforcement in the tensioned zone is: $\sigma_{st} = n_s \sigma_c (1 - k_x - d/h)/k_x \le f_s$

The equilibrium of the vertical forces is:

$$\frac{N}{b\,h} = \tfrac{1}{2} \sigma_c k_x + \frac{\sigma_c \tfrac{1}{2} n_s \omega (2 k_x - 1)}{k_x}$$

Thus the concrete stress is:

$$\sigma_c = \frac{N/(b\,h)}{\tfrac{1}{2} k_x + \tfrac{1}{2} n_s \omega (2 k_x - 1)/k_x} \le f_{c'}$$

The bending moment follows from:

$$\frac{M_e}{b\,h^2} = \tfrac{1}{2} k_x f_c (\tfrac{1}{2} - k_x/3) + \tfrac{1}{2} \omega (\sigma_{st} + \sigma_{sc}) (\tfrac{1}{2} - d/h)$$

If the steel stress exceeds the maximum stress then the forces and bending moments are calculated, assuming the steel stress in the tensioned zone is at maximum, $\sigma_s = f_s$.

The compressive stress acting in the concrete follows from the equilibrium of forces:

$$\sigma_c = \frac{N + \tfrac{1}{2} \omega\, b\, h\, f_s}{\tfrac{1}{2} k_x b\, h + \tfrac{1}{2} n_s b\, h\, \omega (k_x - d/h)/k_x} < f_{c'}$$

The bending moment follows from:

$$\frac{M_e}{b\,h^2} = \tfrac{1}{2} k_x f_c (\tfrac{1}{2} - k_x/3) + \tfrac{1}{2} \omega (f_s + \sigma_{sc}) (\tfrac{1}{2} - d/h)$$

The compressive zone must be enlarged if the concrete stress exceeds the maximum stress.

Provided the stress in the reinforcement does not exceed the maximum stress, the minimum compressive zone follows from the equilibrium of forces, with $\sigma_c = f_c$ and $\sigma_{st} < f_{st}$:

$$\tfrac{1}{2} k_{x\,min}^{2} + \tfrac{1}{2} n_s \omega (2 k_{xmin} - 1) - \frac{N k_{x\,min}}{b\,h\,f_c} = 0$$

The coefficient $k_{x\,min}$ follows from:

$$k_{xmin} = -(n_s\omega - \frac{n_s N}{b\,h\,f_c}) +/- [(n_s\omega - \frac{n_s N}{b\,h\,f_c})^2 + n\omega]^{1/2}$$

This expression can be used if the stress in the reinforcement does not exceed the maximum stress and the compressive zone is smaller than the flange, so the compressive zone has to fulfil the following conditions:

$$\frac{n_s \sigma_c (1 - d/h)}{f_s + n\,\sigma_c} \le k_{x\,min} \le c_f/h$$

The bending moment follows from:

$$\frac{M_e}{b\,h^2} = \tfrac{1}{2} k_x f_c (\tfrac{1}{2} - k_x/3) + \tfrac{1}{2} \omega (\sigma_{st} + \sigma_{sc}) (\tfrac{1}{2} - d/h)$$

Next the curvature and stiffness is calculated as showed before.

Stiffness of the cracked section, serviceability state

For the given section with A = 1000 × 130 mm^2 the stiffness is defined for several values of k_x with $k = \varepsilon_c/x$ for the serviceability state. The vault is tensioned by the time dependent effects. Probably the structure will be cracked, thus for the concrete the tensile stresses are neglected.

Reinforcement Fe220:	$d/h = 0.15$; $E_s = 200000$ MPa; $f_s = 220$ MPa; $\omega = A_t/(b\,h) = 0.0043$;
Concrete C12/15:	$f_c = 12$ MPa; $E = 27000$ MPa;
	$n_s = E_s/E_c = 200000/27000 = 7.4$; $n_s\,\omega = 0.032$

Due to the permanent and asymmetrical load q = 0.5 kN/m the normal load is equal to N = 54 kN and the bending moment is equal to M = 3.1 kNm.

k_x =	$\kappa \times 10^{-6}$ [1/mm]	M [kNm]	EI $\times 10^{12}$ [Nmm²]
1	0.299	1.490	4.991
0.731	0.540	2.219	4.109
0.5	1.045	2.730	2.614
0.385	1.738	3.205	1.844
0.337	2.323	3.564	1.534
0.300	3.070	4.000	1.303
0.269	4.095	4.580	1.118
0.195	12.836	9.308	0.725
0.177	15.758	9.407	0.597

TABLE 5.6 Moment and curvature for the vault

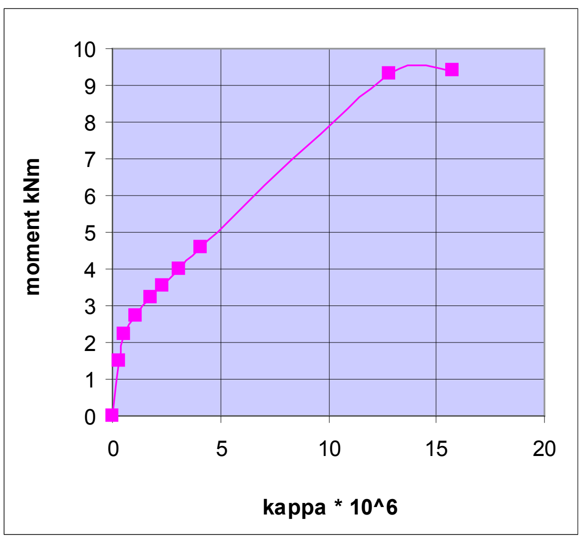

FIGURE 5.16 M-N-k diagram serviceability state, for a normal force equal to N =54 kN

Table 5.6 gives for the normal load, N_{rep} = 54 kN, the bending moments, curvatures and stiffness for several values of the compressive zone of the section with a length $x = k_x h$. The MNκ-diagram shows the curvature with respect to the bending moment.

For M = 3.1 kNm the curvature is defined by interpolating between respectively:

$M = 2.730 \times 10^6$ Nmm; $\kappa = 1.045 \times 10^{-6}$ mm^{-1}
$M = 3.205 \times 10^6$ Nmm; $\kappa = 1.738 \times 10^{-6}$ mm^{-1}

$$\kappa = 1.045 \times 10^{-6} + (1.738 - 1.045) \times 10{-6} \times \frac{n\,(3.1 - 2.73)}{3.205 - 2.73} = 1.585 \times 10^{-6} \text{ mm-1}$$

The stiffness follows from EI = M/κ.

$EI = M/\kappa = 3.1 \times 10^6/(1.585 \times 10^{-6}) = 1.96 \times 10^{12}$ Nmm^2

Buckling

The critical buckling force is calculated with expression [3.13] for the parabolic vault with a length s = 10.31 m and a stiffness equal to $EI = 1.96 \times 10^{12}$ Nmm^2.

$$N_{cr\,asym} = \frac{\pi^2 EI}{s^2} = \frac{\pi^2 \times 1.96 \times 10^{12}}{(10.31 \times 10^3)^2} = 182 \times 10^3 \text{ N}$$

Asymmetrical load

For x = ½ × a the normal force is equal to N_{rep} = 54 kN. Then the ratio n_{cr} of the buckling force and the normal force is equal to: $n_{cr} = N_{cr}/N = 182/54 = 3.4$ The ratio n_{cr} is smaller than the ration n_{cr} calculated before according to the methods practiced fifty years ago. Generally a ratio of $n_{cr} \geq 5$ was recommended.

Due to the second order the bending moment M = 3.1 kNm increases:

$M = 3.1 \times 3.4/(3.4 - 1) = 4.4$ kNm .

The second order effect increases the bending moments substantially.

§ 5.7 Ultimate bearing capacity

Nowadays the reinforcement has to be calculated according to the Eurocode 2 [C6]. The section is not massive, due to the Fusee Céramique elements the section is reduced with a number of hollow cores. The calculation of the required reinforcement is based on a non-linearly stress-strain diagram of the concrete and steel. The cover on the hollow core is c_f. For a compressive zone $k_x h$ larger than c_f the compressive zone has to be reduced with fusées.

Features of steel

The stress in the steel reinforcement must be less than the ultimate stress $\sigma_\sigma \leq f_{yd}$; The stress-strain diagram of the steel is bi-linear: for $\varepsilon_s \leq f_{yd}/E_s$ the maximum stress follows from $\sigma_s = \varepsilon_s E_s$; and for $f_{yd}/E_s < \varepsilon_s < \varepsilon_{su}$ the maximum stress is equal to f_{yd}, f_{yd} is the design load with: $f_{yd} = f_{yk}/\gamma_s$ and γ_s = 1.15.

Features of concrete

The quality of concrete is described with two numbers, the first one mentions the cylindrical strength, the second one the strength of cubes. For C12/15 the cylindrical strength is equal to 12 MPa and the strength of the cubes is equal to 15 MPa. The compressive stress in the concrete must be less than the ultimate stress: $s_c \leq f_{cd}$; with $f_{cd} = f_{ck}/\gamma_c$,where f_{ck} is the cylindrical strength. Generally the safety factor is equal to: $\gamma_c = 1.5$. Thus for C12/15 the ultimate stress is equal to $f_{cd} = f_{ck}/\gamma_c = 12/1.5 = 8$ MPa. The strain is at maximum $\varepsilon_{cu} = 0.0035$. For the ultimate state the stiffness of the concrete is equal to $E_{cd} = f_{cd} /(0.5 \times \varepsilon_{cu})$. For C12/15 Young's modulus is equal to $E_{cd} = 8/(0.00175) = 4571$ MPa.
In the past the stress-strain diagram was schematised parabolic or bilinear. According to the Euro- code the stress-strain diagram of the concrete is schemed rectangularly with $\sigma_c = 0$ for $\varepsilon_c < 0.2 \times 0.0035$ and $\sigma_c = f_c$ for $\varepsilon_c \geq 0.2 \times 0.0035$. The height of the compressed zone $x = k_x h$ is reduced with a factor $\beta = 0.8$. Thus the normal compressive component F_c acting at the compressed side of the concrete follows from:

$$F_c = \beta\, b\, k_x h\, f_{cd} = 0.8 \times k_x b\, h\, f_{cd} \qquad [5.14]$$

Due to the fusées the normal compressive component is reduced with a force F_f following from:

$$F_f = m\, A_f\, f_{cd}$$

With: m is the number of fusées
A_f is the area of the fusées within the compressive zone $\beta\, k_x\, h$

The specific deformation of the steel at the tensioned and compressed side of the section follows respectively from:

$$\varepsilon_{st} = (1 - d/h - k_x)/k_x \quad \text{and} \quad \varepsilon_{sc} = (k_x - d/h)/k_x$$

With: d = distance from the centre of reinforcement to the nearest side.

For $\varepsilon_s < f_{yd}/E_s$ the stress in the steel reinforcement follows from: $\sigma_s = \varepsilon_s E_s$
For $\varepsilon_s > f_{yd}/E_s$ the stress in the steel reinforcement is equal to f_{yd}: $\sigma_s = f_{yd}$

Where f_{yd} is the design load with: $f_{yd} = f_{yk}/\gamma_s$ and $\gamma_s = 1.15$.

The sections of the vault are subjected to a normal force N_d acting eccentric at a distance e_t from the centre. The reinforcement is placed symmetrically in the section. The ultimate normal force and bending moment follows from respectively:

$$N_d = F_c - F_{cf} + F_{sc} - F_{st} \qquad [5.15]$$

$$M_d = (F_{sc} + F_{st}) \times (½\, h - d) + F_c\,(½\, h - 0.4\, k_x\, h) - F_f\, z \qquad [5.16]$$

With:

$F_{st} = F_{sc} = ½\, A_s\, \sigma_s$

For:	$k_x \leq 1.25\ c_f/h$;	$A_f = 0$	$F_f = 0$	$z/r = 0$
	$k_x = 1.25\,(½\, r + c_f)/h$:	$A_f = 0.614\, r^2$	$F_f = m\ 0.614\, r^2 f_{cd}$	$z/r = 0.705$
	$k_x = 1.25\,(r + c_f)/h$:	$A_f = ½\, \pi\, r^2$	$F_f = m\ ½\, \pi\, r^2 f_{cd}$	$z/r = 0.424$
	$k_x \geq 1.25\,(1.5\, r + c_f)/h$:	$A_f = 2.527\, r^2$	$F_f = m\ 2.527\, r^2 f_{cd}$	$z/r = 0.171$
	$k_x \geq 1.25\,(2\, r + c_f)/h$:	$A_f = \pi\, r^2$	$F_f = m\, \pi\, r^2 f_{cd}$	$z/r = 0$

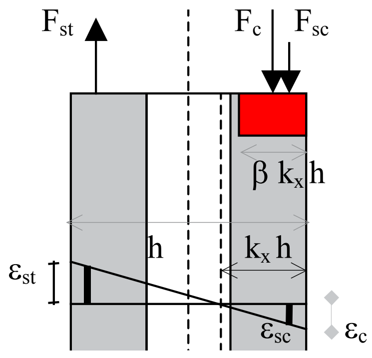

FIGURE 5.17 The stresses and forces acting on a section subjected to an eccentric normal force

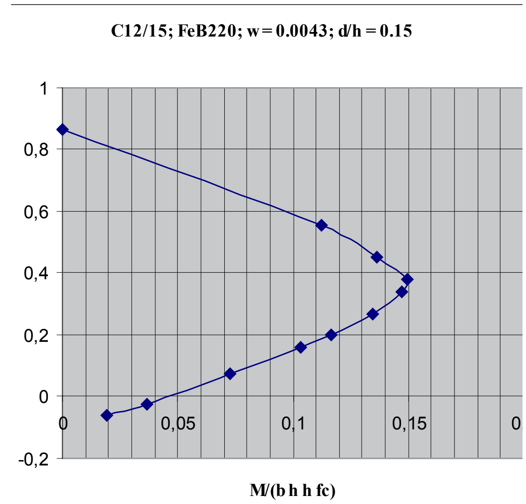

FIGURE 5.18 Graph, the bearing capacity of the vault for: C12/15; Fe220; d/h = 0.15; ω = 0.0043.

$N_d/(b\,h\,f_{cd})$	$Md/(b\,h^2\,f_{cd})$
-0.0627	0.0192
-0.0227	0.0368
0.0735	0.0727
0.1592	0.1033
0.2000	0.1164
0.2692	0.1347
0.3378	0.1471
0.3804	0.1496
0.4536	0.1362
0.5551	0.1121
0.6396	0.0885

TABLE 5.7 Bearing capacity vault C12/15; Fe220; d/h = 0.15; ω = 0.0043

The calculation of an element subjected to a normal force and bending moment column is quite labour intensive, most engineers will use diagrams or spreadsheets to calculate the loadbearing capacity of a section. The sections of the vault are subjected to a normal force N_d acting eccentric at a distance e_t from the centre. The reinforcement is placed symmetrically in the section.

For a structure with C12/15, Fe220 and d/h = 0.15 and a reinforcement $\omega = A_s/(b\,h) = 0.0043$ = 0.43% table 5.7 shows the bearing capacity with on the vertical and horizontal axis respectively:

$$\frac{N_d}{b\,h\,f_{cd}} \quad \text{and} \quad \frac{M_d}{b\,h^2 f_{cd}}$$

For the ultimate limit state the permanent and live loads are increased with a load factor of respectively 1.2 and 1.5.

The permanent load is: $q_{gd} = 1.2 \times 2.4 = 2.9$ kN/m.

The live load is: $q_{ed} = 1.5 \times 0.5 = 0.75$ kN/m.

The vertical and horizontal reaction force acting on the supports are respectively:

$V_{Ad} = q_g a + ¼ q_e a = 2.9 \times 9.9 + ¼ \times 0.75 \times 9.9 = 30.6$ kN

$V_{Bd} = q_g a + ¾ q_e a = 2.9 \times 9.9 + ¾ \times 0.75 \times 9.9 = 34.3$ kN

$$H_d = \frac{q_g a^2}{2f} + \frac{q_e a^2}{4f} = \frac{2.9 \times 9.9^2}{2 \times 2.48} + \frac{0.75 \times 9.9^2}{4 \times 2.48} = 64.7 \text{ kN}$$

The bending exclusive second order moment is equal to:

$M_d = q_e a^2 /16 = 0.75 \times 9.92 /16 = 4.6$ kN

For x = ½ a = 4.95 the normal force follows from:

$N_d = [H_d^2 + (V_{Ad} - q_d x)^2]^{0.5} = [64.7^2 + (30.6 - 2.9 \times 4.95)^2]^{0.5} = 66.7$ kN

Table 5.7 gives for the normal force the ultimate bending moment, with $f_{cd} = 12/1.5 = 8.0$ MPa:

$$\frac{N_d}{b\,h\,f_{cd}} = 0.064 \quad \rightarrow \quad \frac{M_d}{b\,h^2 f_{cd}} = 0.069$$

The ultimate moment the section can resist is equal to $M_u = 9.3$ kNm.

Stiffness of the vault with embedded fusée element for the ultimate state

The stiffness of a structure of concrete can be calculated with a MN-κ diagram. Probably the vault is cracked; for a cracked structure the stiffness can be calculated with expression [5.10]: $EI = M_e/\kappa_e$. Where M_e is the bending moment if the reinforcement reaches the maximum stress. The curvature (kappa) k of a structure loaded by a bending moment and a normal force follows from [5.9]:

$$\kappa = \frac{\sigma_c}{E_c k_x h} \qquad [5.9]$$

Due to the fusees the section of a Fusée Céramique vault is not massive. The compressive zone of a section of a Fusée Céramique vault can be smaller or larger than the compressed flange with a thickness c_f. The bending moment M_e is defined for $k_x < c_f/h$ and $k_x > c_f/h$. Again the curvature and stiffness for the maximum bending bending moment is defined with the procedure showed before for serviceability state.

For the given section with A = 1000 × 130 mm² the features of the section for the ultimate state are: Reinforcement FeB220: d/h = 0.15; E_s = 200000 MPa; f_s = 220/1.15= 191 N/mm²; $\omega = A_t/(b\,h)$ = 0.0043; Concrete C12/15: f_c = 12/1.5 = 8.0 MPa; E_c = 27000/1.2 = 22500 MPa;

For a concrete structure subjected to a permanent compressive load the instantaneous specific deformation is ε. Due to creep the specific deformation will increase with ϕ ε. The total deformation of the concrete is equal to $\varepsilon_o (1 + \phi)$, with ϕ = 4.0:

$E_{ct} = E_c/(1+\phi) = 22500/(1+\phi) = 4500$ MPa

Actually the live load is acting on the roof during a short time. To include the creep Vis and Sagel [Vis91] advises to define the stiffness for a reduced bending moment, $M = 0.8\, M_d$, with a Young's modulus following from the stress-strain diagram: $E_{ct} = f_c/\varepsilon_c$ with $\varepsilon_c = 0.00175$.

For C12/15: E_{ct} = 8/0.00175 = 4571 MPa.

According to the Euro-code Young's modulus can de calculated with an effective creep factor ϕ_{ef} following from: $\phi_{ef} = \phi_t M_{E\phi}/M_{Ed}$ Due to the permanent load the vault is subjected to a bending moment:

$$M_{E\phi} = {}^2/_{81} \times 0.118 \times q\, a^2 = {}^2/_{81} \times 0.118 \times 2.4 \times 9.9^2 = 0.69 \text{ kNm}$$

The maximum moment due to the asymmetrical load is equal to $M_{Ed} = 4.6$ kNm, substituting this moment into [5.23] gives the creep factor ϕ_{ef}:

$$\phi_{ef} = \phi_t M_{rep}/M_{Ed} = 4.0 \times 0.69/4.6 = 0.6$$

For the instantaneous load Young's modulus follows from:

$$E_{ct} = E_{cd}/(1+\phi_{ef}) = 22500/(1+0.6) = 14063 \text{ MPa}$$

For the permanent and the asymmetrical live load the normal force and bending moment are respectively $N_d = 66.7$ kN and $M_d = 4.6$ kNm. The following diagram and table show for the vault subjected to this load the stiffness.

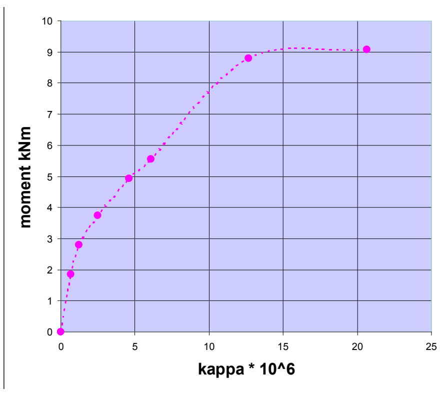

FIGURE 5.19 MN-κ diagram for N =66.7 kN (UGT)

k_x =	kappa × 10^6 [1/mm]	M [kNm]	EI × 10^{12} [Nmm²]
Infinitive	0	0	
1	0.68	1.85	2.713
0.731	1.24	2.80	2.253
0.5	2.48	3.75	1.512
0.385	4.59	4.93	1.074
0.337	6.08	5.55	0.912
0.269	12.68	8.80	0.694
0.212	20.64	9.07	0.439

TABLE 5.8 Stiffness of the vault, UGT

For M = 4.6 kNm the stiffness is defined by interpolating between respectively:

$$M = 3.75 \times 10^6 \text{ Nmm}; \quad \kappa = 2.48 \times 10^{-6} \text{ mm}^{-1}$$

$$M = 4.93 \times 10^6 \text{ Nmm}; \quad \kappa = 4.59 \times 10^{-6} \text{ mm}^{-1}$$

$$\kappa = 2.48 \times 10^{-6} + (4.59-2.48) \times 10^{-6} \times \frac{(4.6 - 3.75)}{4.93 - 3.75} = 4.0 \times 10^{-6} \text{ mm}^{-1}$$

$$EI_{ct} = M_d/\kappa = 4.6 \times 10^6/(4.0 \times 10^{-6}) = 1.15 \times 10^{12} \text{ Nmm}^2$$

Buckling ultimate state

For the vault with a length s = 10.31 m, a stiffness equal to EI = 1.18×10^{12} Nmm^2, and $\psi = 1$ the critical buckling force is:

$$N_{cr} = \frac{\pi^2 EI}{(\psi s)^2} = \frac{\pi^2 \times 1.15 \times 10^{12}}{(10.31 \times 10^3)^2} = 106.8 \times 10^3 \text{ N}$$

For x = ½ a the normal force due to the permanent an asymmetric live load is equal to N_d = 66.7 kN. Thus the ratio n of the buckling force and the normal force is equal to:$n_{cr} = N_{cr}/N = 106.8/66.7 = 1.6$
The ratio n_{cr} is very small so the increase due to the second order is substantially. The bending moment inclusive of second order effects is:

$$M_d = 4.6 \times 1.6/(1.6 - 1) = 12.3 \text{ kNm} > M_u \quad (\text{with } M_u = 9.3 \text{ kNm})$$

For the calculation of the buckling length the effect of hangers was neglected. The following analysis shows the effect of the hangers for the buckling load. The reduction of the buckling load due to the hangers between the tie and vault is described in chapter 3. For a vault with three hangers with one hanger tensioned due to the upward deformation the factor ψ follows from [3.56]:

$$\psi = [1 - {}^1/_6 \cos\alpha]^{1/2}.$$

For f/*l* = 0.125 and cos α = 0.894 the reduction factor ψ is: $\psi = (1 - 0.894/6)^{1/2} = 0.922$.
For the vault with a length s = 10.31 m and a stiffness EI = 1.18×10^{12} Nmm^2, the critical buckling force is:

$$N_{cr} = \frac{\pi^2 EI}{(\psi s)^2} = \frac{\pi^2 \times 1.15 \times 10^{12}}{(0.922 \times 10.31 \times 10^3)^2} = 125.6 \times 10^3 \text{ N}$$

Thus the ratio n_{cr} of the buckling force and the normal force is equal to:$n_{cr} = N_{cr}/N = 125.6/66.7 = 1.9$
The ratio n_{cr} is very small, in practice it is recommended to design a structure so $n_{cr} \geq 5$. The increase due to the second order is substantially. The bending moment inclusive of second order effects is:

$$M_d = 4.6 \times 1.9/(1.9 - 1) = 9.7 \text{ kNm} > M_u$$

The structure does not meet the demands of the present. To be safe the structure must be strengthened and stiffened. Chapter 6 shows techniques to strengthen and stiffen vaults and chapter 7 shows the effect of the strengthening for this vault.

6 Strengthening parabolic vaults

Introduction

The sections of arches and vaults subjected to varying load combinations are often simultaneously loaded by normal forces and bending moments. To resist the bending moments the thickness of these arches and vaults must be increased or these structures need to be strengthened. Strengthening arches and vaults with diagonals helps to decrease the bending moments; consequently these strengthened arches and vaults can be designed with a very low thickness. Increasing the number of the hangers connecting the arches and vaults with the ties increase the resistance. Palkowski [Pal12] showed for parabolic arches the increase of the critical buckling load due to the effect of ties and hangers. The number of the hangers as well as the position and inclination affects the critical buckling load. Curving the ties, between the supports, slightly upward helps to increase the load bearing capacity too. In the past the Russian engineer V.G.Shukhov strengthened the half circular arches of the Gum warehouse in Moscow [Bel77] with cables running from the supports to varying points at the arches dividing the arches in segments, as is shown in figure 6.1. These cables prevent the sideward deformation of the a rches, which is caused by both horizontal and asymmetrical loads. A half circular arch is curved more than a parabola or a catenary. Consequently the system line of a half circular arch is positioned outward from the line of the thrust. Due to the permanent load the structure is bending outward, this deformation will be explained in chapter 9. The cables running from the supports to the arch prevent the arch deforming outward and reduce the bending moments due to the permanent load considerably. Thus the cables running from the supports to the arch are very effective for a half circular vault, but are less effective for a deep parabolic vault.

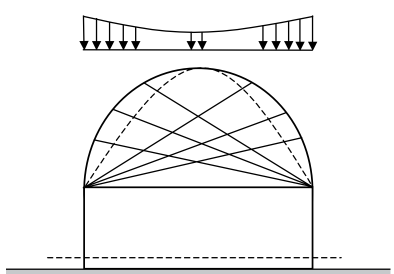

FIGURE 6.1 Half circular arch strengthened with pre-stressed cables.

For shallow parabolic arches and vaults diagonals running from the supports to the top decrease the bending moments caused by asymmetric loads considerably and increase the critical buckling load. This chapter analyses the effect of the diagonals for these shallow parabolic curved vaults under varying load conditions, such as for example an equally distributed load, a surface load and an antimetric load. To show the effect of the diagonals the bending moments and normal forces are defined firstly for a non-strengthened vault and later as well for the strengthened vault. The expressions defined for the normal forces and bending moments are very useful for the preliminary design. Varying parameters shows the effect of changing curvature, rise and dimensions. The bending moments are described according to the Theory of Elasticity. In practice arches are designed with two hinged supports, as well as two hinged supports and an extra hinge at the top.

The analysis of a statically determinate structure is much easier than the analysis of a statically indeterminate structure. Firstly the statically determinate structures are analysed for parabolic vaults. Later it will be showed that the results defined for the statically determinate vaults can be used for the statically indeterminate structures as well.

§ 6.1 A general description of the features of parabolic vaults

For a vault following a parabola the centre of the coordinates is placed on the top, see figure 6.2. Assume the half of the length is equal to a so the span is equal to 2 a. The parabola is described with:

$$y = f x^2/a^2 \qquad [6.1]$$

Differentiation gives the inclination of the tangent: $y' = dy/dx = 2 f x/a^2$ [6.2]

The second derivative of the parabolic curve is constant: $y'' = 2 f/a^2$ [6.3]

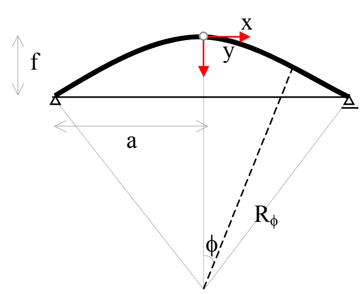

FIGURE 6.2 The parabolic vault, the centre of the X and Y-axis is positioned at the top

The length of a parabola

The length of the parabola can be defined mathematically as follows. The length of a small part of the parabolic vault, ds, is equal to: $ds = dx\,(1+ dy^2/dx^2)^{1/2}$. The length of the vault is defined by integrating ds: $s = \int ds$. Substituting dy/dx [6.2] gives:

$$s = \int (1+ 4 f^2 x^2/a^4)^{1/2} dx$$

This expression can be simplified with: $s = 2 f/a^2 \times \int (\tfrac{1}{4} a^4/f^2 + x^2)^{1/2} dx$

According to Sherwood et al [She58] this expression is integrated for the interval between x = 0 and x = a:

$$s = 2 f/a^2 \times \tfrac{1}{2} \times [\, x\,(x^2 + \tfrac{1}{4} a^4/f^2)^{1/2} + \tfrac{1}{4} a^4/f^2 \times \ln\{x + [x^2+ \tfrac{1}{4} a^4/f^2]^{1/2}\}\,]_{x=0}^{x=a}$$

$$s = f/a^2\, [\, a\,(a^2+ \tfrac{1}{4} a^4/f^2)^{1/2} + \tfrac{1}{4} a^4/f^2 \times \ln\{a +(a^2+ \tfrac{1}{4} a^4/f^2)^{1/2}\} - \tfrac{1}{4} a^4/f^2 \times \ln(\tfrac{1}{2} a^2/f)\,]$$

$$s = f\,(1+ \tfrac{1}{4} a^2/f^2)^{1/2} + \tfrac{1}{4} a^2/f \times \ln\{a + (a^2 + \tfrac{1}{4} a^4/f^2)^{1/2}\} - \tfrac{1}{4} a^2/f \times \ln(\tfrac{1}{2} a^2/f)$$

$$s = f\,(1+ \tfrac{1}{4} a^2/f^2)^{1/2} + \tfrac{1}{4} a^2/f \times \ln\{2 f/a + (4 f^2/a^2+1)^{1/2}\} \qquad [6.4]$$

For f = ¼ a the length s is equal to 1.04 a.

The radius of the parabolic vault

The curvature of the vault is equal to 1/R; the radius of a parabola varies, so the curvature of the parabola also varies. A small part of the length of the arch ds is defined with: $ds = R_\phi d\phi$. This expression can be written as: $R_\phi = (ds/dx) \times (dx/d\phi)$

The variation of $d\phi$ over a small piece with length dx is equal to: $d\phi/dx = d(dy/dx)/dx = y''$.

Substituting ds and $d\phi/dx$ into the expression for the radius R_ϕ gives: $R_\phi = \dfrac{(1+y'^2)^{1/2}}{y''}$

Substituting y' and y'' into this expression gives:

$$R_\phi = \frac{(1+4f^2x^2/a^4)^{1/2}}{2f/a^2} \qquad [6.5]$$

For x = 0 and for x = a the radius is respectively equal to:

$$R_{\phi=0} = \tfrac{1}{2}a^2/f \quad \text{and} \quad R_{x=a} = \tfrac{1}{2}a^2/f \times (1+4f^2/a^2)^{1/2} \qquad [6.6]$$

For a parabolic vault the radius R_ϕ inc reases for an increasing value of x.

§ 6.2 Three hinged parabolic vault

For a statically determinate three hinged parabolic vault the forces and bending moments are defined for several loads.

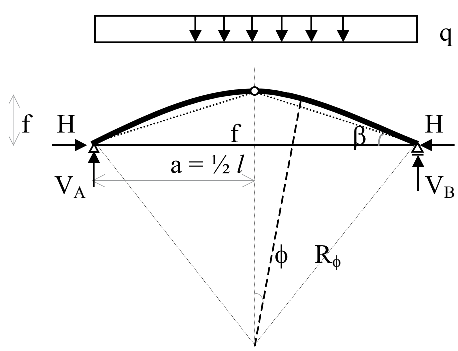

FIGURE 6.3 Three hinged parabolic vault subjected to a symmetrically equally distributed load q

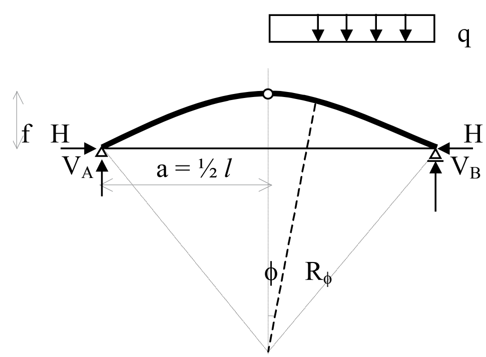

FIGURE 6.4 Three hinged parabolic vault subjected to an asymmetrically equally distributed load q

Three hinged parabolic vault subjected to an equally distributed load

Firstly the load transfer is analysed for a parabolic vault, which is simply supported and hinged at the top. The structure is statically determinate. The forces and bending moments are defined for a symmetric and asymmetrical load which is equally distributed over the length of the arch. The parabola with the centre of the coordinates in the top is described again with: $y = f x^2/a^2$.

The vault is subjected to an equally distributed load q. The vertical reactions acting at the supports at the left and right side, V_A and V_B, are equal to: $V_A = V_B = q\ a$. The thrust H, acting at the supports, follows from the equilibrium of the moments around the hinge at the top.

For the right part we find: $M = H \times f - V_B\, a + ½\, q\, a^2 = 0$

Substituting $V_B = q\, a$ into this expression:

$$H = q\, a^2/f - ½\, q\, a^2/f \quad \rightarrow \quad H = ½\, q\, a^2/f \qquad [6.7]$$

Next the bending moments are defined. For the right part the bending moment M_x at a certain distance x_1 from the centre follows from: $M_{x1} = H\, y_1 - \int q\, (x_1 - x)\, dx$

Substituting $y_1 = f\, x_1^2/a^2$ [6.1] and $H = ½\, q\, a^2/f$ [6.7] into this expression and integrate between $x = 0$ and $x = x_1$:

$$M_{x1} = ½\, q\, a^2/f \times f\, x_1^2/a^2 - q\, x_1^2 + ½\, q\, x_1^2 \rightarrow \quad M_{x1} = 0$$

For any value of $x = x_1$ the bending moment is zero. The parabolic vault subjected to an equally distributed load is not loaded in bending. For this symmetrically loaded arch the same result is found for the left part. This vault is subjected to normal forces N_x only. The normal forces can be calculated with: $N_x = (H^2 + V_x^2)^{½}$

Three hinged parabolic vault subjected to an asymmetrically equally distributed load

The three hinged parabolic vault will be subjected to bending if the vault is loaded asymmetrically. Assume the vault is simple supported, hinged at the top and subjected to an equally distributed load q acting al the right part. The structure is statically determinate.

The vertical reaction acting at the support at the right side, V_B, is defined with the equilibrium of the bending moments around the support at the left side:

$$V_B \times 2\, a = q\, a \times {}^3/_2\, a \quad \rightarrow \quad V_B = ¾\, q\, a$$

The reaction V_A at the support at the left side follows from the equilibrium of the vertical load and reactions:

$$V_B + V_B = q\, a;\ V_B = ¾\, q\, a \quad \rightarrow \quad V_A = ¼\, q\, a$$

The thrust H acting at the supports follows from the equilibrium of the moments around the hinge at the top. For the left part we find:

$$H\, f - V_A\, a = 0 \quad \rightarrow \quad H = ¼\, q\, a^2/f \qquad [6.8]$$

For the right part of the arch the thrust H follows from the equilibrium of the moments around the hinge at the top too. $H\, f - V_B\, a + ½\, q\, a^2 = 0$

To define H substituting $V_B = ¾\, q_e\, a$ into this expression: $H = ¾\, q\, a^2/f - ½\, q\, a^2/f$

Again the thrust is equal to: $H = ¼\, q\, a^2/f$

For the vault subjected to a vertical load the thrust does not vary. For the left part the bending moment M_x at a distance x from the support at the left side follows from:

$$M_x = H\, y - ¼\, q\, a\, x$$

Substituting $H = ¼\, q\, a^2/f$ and $y = f\, x^2/a^2$ gives:

$$M_x = (¼\, q\, a^2/f) \times f\, x^2/a^2 - ¼\, q\, a\, x \quad \rightarrow \quad M_x = ¼\, q\, x^2 - ¼\, q\, a\, x \qquad [6.9]$$

The bending moment is at maximum for $dM_x/dx = 0$. Differentiating expression (6.9) gives:

$$dM_x/dx = ½\, q\, x - ¼\, q\, a = 0 \quad \rightarrow \quad x = ½\, a$$

So the bending moment is at maximum for x = ½ a. To define the maximum moment x = ½ a is substituted into the expression for M_x :

$M_{x\,max}$ = ¼ q (½ a)² - ¼ q a × ½ a → $M_{x\,max}$ = - $^1/_{16}$ q a² [6.10]

Due to this bending moment the arch will be tensioned at the outer side.

For the right part the bending moment is equal to: M_x = ¼ q a x + H y - ½ q x²
Substituting the thrust H = ¼ q a²/f into this expression:

M_x = ¼ q a x + (¼ q a²/f) × f x²/a² - ½ q x² → M_x = ¼ q a x − ¼ q x²

This expression is identical to the expression defined for the bending moment at the left part of the arch [6.9]. Again the maximum moment is found for x = ½ a. Evidently for the right part the maximum moment is equal to the maximum moment defined for the left part of the vault [6.10], thus: $M_{x\,max}$ = $^1/_{16}$ q a² . Due to this bending moment the arch will be tensioned at the inner side.

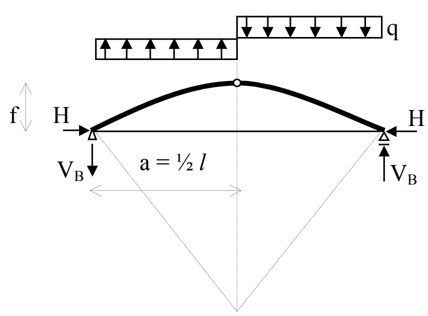

FIGURE 6.5 Parabolic vault subjected to an anti-metrically load q

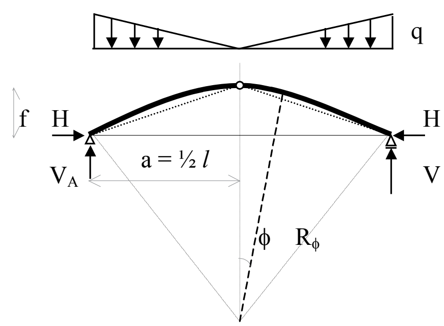

FIGURE 6.6 Parabolic vault subjected to an increasing load q

Three hinged parabolic vault subjected to an anti-metrically load

An asymmetrically load q acting at one side can be considered as the combination of an equally distributed load equal to ½ q and an anti-metrically load. ½ q. For a vault subjected to an antimetrical load q the bending moments and normal forces will be defined. Assume the vault is simple supported, hinged at the top and subjected to an equally distributed load q acting upward al the left part and acting downward at the right part. The structure is statically determinate.
The vertical reaction acting at the support at the right side, V_B, is defined with the equilibrium of the bending moments around the support at the left side:

V_B × 2 a = q a × $^3/_2$ a - q a × ½ a → V_B = ½ q a ↑

The reaction V_A at the support at the left side is equal to: V_A = ½ q a ↓

The thrust H acting at the supports follows from the equilibrium of the moments around the top.
For the right part we find: H × f - V_B a + ½ q a² = 0

To define H substituting V_B = ¾ q a into this expression: H = ½ q a²/f − ½ q a²/f = 0 [6.11]

For the right part the bending moment M_x at a distance x from the support at the left side follows from:

M_x = H y + ½ q a x - ½ q x²

Substituting $H = 0$ and $y = f\,x^2/a^2$ gives: $M_x = \frac{1}{2} q\, a^2 (x/a - x^2/a^2)$ [6.12]

The bending moment is at maximum for $dM_x/dx = 0$. Differentiating expression [6.9] gives:

$$dM_x/dx = \tfrac{1}{2} q\, a^2 \,(1/a - 2x/a^2) = 0 \quad \rightarrow \quad x = \tfrac{1}{2} a$$

So the bending moment is at maximum for $x = \frac{1}{2} a$. To define the maximum moment $x = \frac{1}{2} a$ is substituted into the expression for M_x:

$$M_{x\,max} = \tfrac{1}{2} q\, a^2 \times (\tfrac{1}{2} - \tfrac{1}{4}) \quad \rightarrow \quad M_{x\,max} = {}^1/_8\, q\, a^2 \qquad [6.13]$$

Due to this bending moment the vault will be tensioned at the inner side.

For a vault subjected to an asymmetrical load q the structure can be considered as loaded by the combination of an equally distributed load ½ q and an antimetically load ½ q. For this load combination the thrust H and bending moment M is equal to respectively:

$$H = \tfrac{1}{2} \times (\tfrac{1}{2} q)\, a^2/f + 0 = \tfrac{1}{4} q\, a^2/f \qquad M_{max} = {}^1/_{16}\, q\, a^2$$

Three hinged parabolic vault subjected to a symmetric triangular load

The three hinged parabolic vault is subjected to bending in case this vault is subjected to a load increasing linearly from the centre to the supports. Assume the parabolic vault is simply supported, hinged at the top and subjected to an increasing load q which is zero at the top and maximum at the base q. The structure is statically determinate. The vertical reaction acting at the support A and B is equal to: $V_A = V_B = \frac{1}{2} q\, a$. The thrust H acting at the supports follows from the equilibrium of the moments around the centre at the top.

$$H f - V a + {}^1/_3\, q\, a^2 = 0 \quad \rightarrow \quad H = {}^1/_6\, q\, a^2/f \qquad [6.14]$$

The normal force acting at the vault is at maximum for $x = a$, with: $N_{x=a} = (H^2 + V_{x=a}{}^2)^{1/2}$
Substituting H and V:

$$N_{x=a} = [({}^1/_6\, q\, a^2/f)^2 + (\tfrac{1}{2} q\, a)^2]^{1/2} \quad \rightarrow \quad N = \tfrac{1}{2} q\, a\, [1 + {}^1/_9\, q\, a^2/f^2]^{1/2}$$

The bending moment M_x at a distance x from the top follows from:

$$M_x = H\, y - \tfrac{1}{2} \times {}^1/_3\, q\; x^3/a \quad \rightarrow \quad M_x = ({}^1/_6\, q\, a^2/f) \times f\, x^2/a^2 - {}^1/_6\, q\, x^3/a$$

This expression can be simplified into: $M_x = {}^1/_6\, q\,(x^2 - x^3/a)$ [6.15]

The bending moment is at maximum for $dM_x/dx = 0$. the position of the maximum moment is found by differentiating expression [6.15]:

$$dM_x/dx = {}^1/_6\, q\,(2x - 3x^2/a) = 0 \quad \rightarrow \quad x = 0 \text{ or } x = {}^2/_3\, a$$

The bending moment is at maximum for $x = {}^2/_3\, a$, substituting $x = {}^2/_3\, a$ into expression [6.15] results in:

$$M_x = {}^1/_6\, q_e\, [({}^2/_3 a)^2 - ({}^2/_3 a)^3/a] \quad \rightarrow \quad M_x = {}^2/_{81}\, q\; a^2 \qquad [6.16]$$

Three hinged vault subjected to a surface load

Due to a dead load the vault will be subjected to a force increasing from the top to the supports which is equal to q at the top and maximum at the base with: $q_{max} = q\ ds/dx = q\ (1+(y')^2)^{½}$
Substituting $y' = 2\,f\,x/a^2$ into this expression: $q_{max} = q\ [1 + (2\,f\,x/a^2)^2]^{½}$
At the supports for x = a the maximum load is equal to: $q_{max} = q\ (1+ 4\,f^2/a^2)^{½}$

For a low rise vault the increase of the load is quite small. For f = a/4 the maximum load at the support is equal to $q_{max} = 1.118$ q. To simplify the calculations for low-rise vaults the dead load is assumed to be increasing linearly from q at the top to (1+c) q at the support. For these low-rise vaults the parameter c can be found by assuming for the linearly increasing load the maximum load at the support equal to the maximum load for the dead load, thus:

$$q_{max} = q \times (1+ 4\,f^2/a^2)^{½} = q\,(1 + c) \quad \rightarrow \quad c = (1+ 4\,f^2/a^2)^{½} - 1$$

For f = ¼ a the parameter c is equal to 0.118. For the three-hinged parabolic arch subjected to a linearly increasing load the bending moments are defined as follows. The structure is statically determinate. Due to the triangular load the structure is subjected to a normal force and a bending moment.

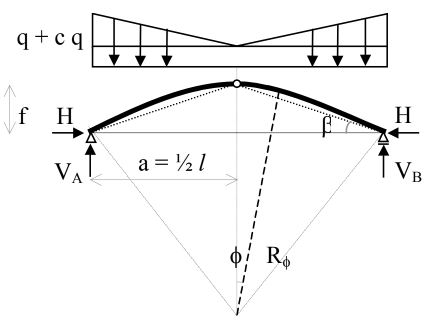

FIGURE 6.7 Parabolic vault subjected to a equally distributed load q and an increasing load c q

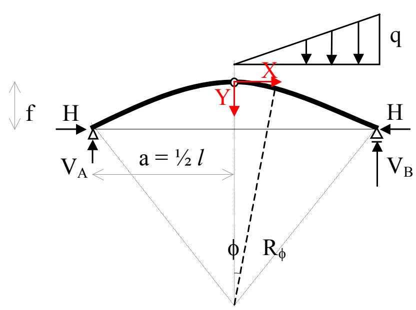

FIGURE 6.8 Parabolic vault, subjected to an asymmetrical load q increasing from the centre to the support

The vertical reaction V acting at the support A or B is: $V = q\,a + ½\,c\,q\,a$
The thrust H follows from the equilibrium of bending moments around the top:

$$H\,f - V\,a + ½\,q\,a^2 + {}^1/_3\,c\,q\,a^2 = 0$$

Substituting V:

$$H = (½\ q\,a^2 + {}^1/_6\ c\,q\,a^2)/\,f \quad \rightarrow \quad H = (1+ {}^1/_3\,c\,) \times ½\ q\,a^2/f$$

The maximum normal force acting at the vault is: $N = (H^2 + V^2)^{½}$

Substituting H and V: $N = [\,(½\,q\,a^2/\,f \times (1+ {}^1/_3\,c\,))^2 + (q\,a + ½\,c\,q\,a)^2\,]^{½} \quad \rightarrow$

$$N = q\,a \times \{\,[½\,a/f \times (1 + {}^1/_3\,c)]^2 + (1 + ½\,c)^2\,\}^{½}$$

The bending moment is at maximum for $x = {}^2/_3$ a from the top:

$$M_x = {}^2/_{81}\,c\,q\,a^2 \qquad [6.18]$$

Asymmetric load increasing linearly from the top to one support

The vault is subjected to an asymmetric equally distributed load q acting at a half of the vault. The minimum and maximum vertical reactions are respectively equal to: $V_a = {}^1/_{12}\, q\, a$ and $V_b = {}^5/_{12}\, q\, a$
The thrust is calculated with the equilibrium of moments round the centre. Placing a virtual hinge at the top, means the bending moment acting on the top is zero. For the left side of the vault the equilibrium of bending moments around the support is equal to:

For the left side: $M = H\, f - V_a\, a = 0 \;\rightarrow\; H = {}^1/_{12}\, q\, a^2/f$ [6.19]

The bending moment at a distance x from the top follows from:

$$M_x = H\, y + V_a\, x - ½\, q\, (x/a)\, x^2/3$$

Substituting H, V_a, y, gives: $M_x = {}^1/_{12}\, q\, a^2/f \times f\, x^2/a^2 + {}^1/_{12}\, q\, a\, x - {}^1/_6\, q\, (x/a)\, x^2 \;\rightarrow$

$$M_x = {}^1/_{12}\, q\, x^2 + {}^1/_{12}\, q\, a\, x - {}^1/_5\, q\, x^3/a$$

The position of maximum moment is found by differentiating this expression:

$$\frac{dM_x}{dx} = q\, x/6 + q\, a^2/12 - ½\, q\, x^2 = 0 \quad\rightarrow\quad x/a = (1 + \sqrt{7})/6 = 0.6076$$

To find the maximum moment the value of x is substituted into the expression for the bending moment:

$$M_{x\,max} = 0.044\, q\, a^2$$ [6.20]

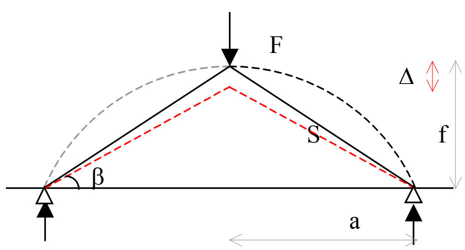

FIGURE 6.9 Parabolic vault, subjected to a concentrated load F. The line of thrust is dotted

Three hinged parabolic vault subjected to a concentrated load acting at the top

The three-hinged parabolic vault will be subjected to bending if this vault is loaded by concentrated load acting vertically at the centre. Assuming the arch is simply supported and hinged at the top. The structure is statically determinate.
The vertical reaction acting at the support at the left V_A, and right side V_B, is equal to: $V_A = V_B = ½\, F$.
The thrust H acting at the supports follows from the equilibrium of the moments around the hinge at the top. For the left part we find:

$H\, f - V_A\, a = 0 \quad\rightarrow\quad H = ½\, F\, a/f$ 21

The bending moment M_x at a distance x from the support at the left side follows from:

$$M_x = H\, y - ½\, F\, x$$

Substituting $H = ½\, F\, a/f$ and $y = f\, x^2/a^2$: $\rightarrow$ $M_x = ½\, F\, a/f \times (f\, x^2/a^2) - ½\, F\, x$

This expression can be simplified into: $M_x = ½ F a (x^2/a^2 - x/a)$ [6.22]

The maximum bending moment is found for $dM_x/dx = 0$. Differentiating the expression for the bending moment:

$$dM_x/dx = ½ F a (2 x/a^2 - 1/a) = 0 \quad \rightarrow \quad x = ½ a$$

The maximum bending moment is found for x = ½ a. Substituting x = ½ a into the expression for M_x gives the maximum moment:

$$M_x = ½ F a (¼ a^2/a^2 - ½ a/a) \quad \rightarrow \quad M_{x\,max} = - F a/8$$

The bending moment is negative, due to the bending moment the vault is tensioned at the outer side.

Three hinged parabolic vault subjected to a concentrated horizontal load acting at the top

The three-hinged parabolic vault will be subjected to bending if this vault is loaded horizontally by concentrated load acting at the centre. Assuming the arch is simply supported and hinged at the top. The structure is statically determinate. Due to the force H acting at the top both supports are subjected to a force ½ H. The vertical reaction forces acting at the supports follow from the equilibrium of the moments around the hinge at the top. For the left part we find:

$$H f - V_A a = 0 \quad \rightarrow \quad V_a = ½ H f/a$$

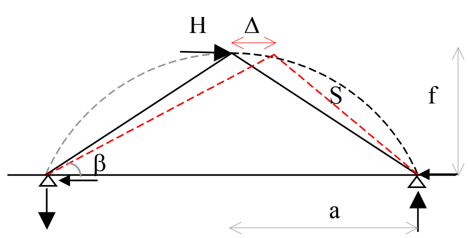

FIGURE 6.10 Parabolic vault, subjected to a concentrated load H. The line of thrust is dotted

The bending moment M_x at a distance x from the support at the left side follows from:

$$M_x = ½ H y - V x$$

Substituting $V = ½ H f/a$ and $y = f x^2/a^2$: $M_x = ½ H f \times (x^2/a^2 - x/a)$ [6.24]

The bending moment is at maximum for $dM_x/dx = 0$. Differentiating the expression for the bending moment:

$$dM_x/dx = ½ H f (2 x/a^2 - 1/a) = 0 \quad \rightarrow \quad x = ½ a$$

So the bending moment is at maximum for x = ½ a. Substituting x = ½ a into the expression for M_x to define the maximum moment:

$$M_x = ½ H f (¼ a^2/a^2 - ½ a/a) \quad \rightarrow \quad M_{x\,max} = - F a/8$$

The bending moment is negative, due to the bending moment the vault is tensioned at the outer side.

§ 6.3 Example, three hinged vault

For a three-hinged vault following a parabola the normal forces and bending moments are defined for several loads. The span is equal to l = 2 a = 14.4 m, the rise is of the swallow vault is equal to f = l/8 = 1.8 m. The structure is composed of concrete fusées and reinforced with steel. In a section with a width of 1.0 m a number of eleven fusées are placed with a spacing of 10 mm. The centre-to-centre distance of the ceramic elements is equal to 90 mm. The thickness of the vault is 110 mm. To resist the thrust, steel bars Ø25 are made at a centre-to-centre distance of 1.0 m. The vault is reinforced with bars Ø8 – 180 at the top and bottom. Distribution bars are not used, the rebars Ø8 – 180, parallel to the span, are positioned between the fusées with a covering of 15 mm at the top and bottom of the vault. According to the calculations made for a factory in Dongen, the Young's modulus of the fusées, concrete C12/15 and reinforcement is respectively equal to $E_f = 1.7 \times 10^4$ MPa, $E_c = 2.7 \times 10^4$ MPa and $E_s = 2.0 \times 10^5$ MPa. For a part of the roof with width of 1.0 m the areas and second moment of the area of the concrete, fusées and steel are calculated:

		Area	
Fusées	A_f =	$11 \times ¼ \pi \times (80^2-60^2)$ =	24.19×10^3 mm^2
Concrete	A_c =	$110 \times 1000 - 11 \times ¼ \pi\ 80^2$ =	54.71×10^3 mm^2
Rebars at the top	A_s =	$¼ \pi \times 8^2 \times 1000/180$ =	279 mm^2
Rebars at the bottom	A_s =	$¼ \pi \times 8^2 \times 1000/180$ =	279 mm^2
		Second moment of the Area	
Concrete	I_c =	$1000 \times 110^3/12 - 11 \times \pi \times 80^4/64$ =	88.80×10^6 mm^4
Fusées	I_f =	$11 \times \pi\ (80^4 - 60^4)/64$ =	15.12×10^6 mm^4
Rebar's 2Ø8-180	I_s =	$2 \times 279 \times (½ \times 110 - 15 - ½ \times 8)^2$ =	0.76×10^6 mm^4

The distribution and transfer of the loads is defined according to the Theory of Elasticity. The stiffness is calculated by multiplying the area, A, and second moment of the area, I, with the Young's modulus. To simplify the calculation a ratio n_f and n_s is introduced with: $n_f = E_f / E_s = 0.63$ and $n_s = E_s / E_c = 7.4$. Thus EA and EI are calculated with respectively:

$$EA = E_c (A_c + n_f A_f + n_s A_s) \quad [6.25]$$

$$EI = E_c (I_c + n_f I_f + n_s I_s) \quad [6.26]$$

Substituting the values for E_c, I_c, I_f and I_s into these equations gives for this vault:

$$EA = 2.7 \times 10^4 \times (54.71\times10^3 + 0.63 \times 24.19\times10^3 + 7.4 \times 2 \times 279) = 2.0 \times 10^9 \text{ N}$$

$$EI = 2.7 \times 10^4 \times (88.80\times10^6 + 0.63 \times 15.12\times10^6 + 7.4 \times 0.72 \times 10^6) = 2.8 \times 10^{12} \text{ Nmm}^2$$

This roof is designed for a live load p = 1.0 kN/m^2. The mass of the concrete and fusées is respectively 2400 kg/m^3 and 1800 kg/m^3. The permanent load is equal to:

Dead weight of the vault: p_g = 0.055 × 24 + 0.024 × 18 =	1.75	kN/m^2
Finishing and ceiling:	0.25	kN/m^2
Permanent load:	2.0	kN/m^2

Permanent load

Due to the dead load the vault is subjected to a surface load equal to: q_g = 2.0 kN/m^2. At the supports for x = a the maximum load is equal to: $q_{max} = q\,(1+ 4\,f^2/a^2)^{½}$

To simplify the calculations for low rise vaults the dead load is assumed to be increasing linearly from q at the top to (1+c) q at the support. For these low-rise vaults the parameter c can be found by assuming for the linearly increasing load the maximum load at the support equal to the maximum load for the dead load, thus:

$$q_{max} = q\,(1+ 4\,f^2/a^2)^{½} = q\,(1 + c) \quad \rightarrow \quad c = (1+ 4\,f^2/a^2)^{½} - 1$$

For f = ¼ a the parameter c is equal to 0.118.

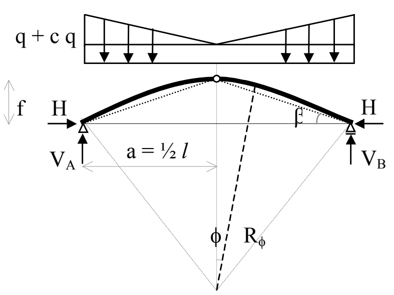

FIGURE 6.11 Parabolic vault subjected to an equally distributed load q and an increasing load c q

The vertical reaction V acting at the support A or B is equal to:

$$V = q\,a + ½\,c\,q\,a = 2.0 \times 7.2 \times (1.0 + 0.118/2) = 15.3 \text{ kN}$$

The thrust H follows from expression (6.17): $H = (1+ {}^1/_3\,c) \times ½\,q\,a^2/ f = 29.9$ kN

For x = ½ a the vertical force V_x is equal to:

$$V_{x = ½ a} = ½\,q\,a + ½\,c\,q \times ½\,a \times ½ = ½\,q\,a\,[1 + ¼\,c] = ½ \times 2.0 \times 7.2 \times [1 + ¼ \times 0.118] = 7.4 \text{ kN}$$

The bending moment is at maximum for x = ${}^2/_3$ a from the top, the maximum bending moment follows from expression (6.18):

$$M_{max} = {}^2/_{81}\,c\,q\,a^2 = {}^2/_{81} \times 0.118 \times 2.0 \times 7.2^2 = 0.3 \text{ kNm}$$

For x = ½ a this bending moment is a litle bit smaller than the maximum moment. The maximum bending moment follows from expression (6.15): $M_x = {}^1/_6\,q\,(x^2 - x^3/a)$ [6.15]

Sunstituting x = ½ a gives:

$$M_{max} = {}^1/_{48}\,c\,q\,a^2 = {}^1/_{48} \times 0.118 \times 2.0 \times 7.2^2 = 0.25 \text{ kNm}$$

These bending moments are much smaller than the maximum bending moment due to an asymmetrical live load.
The normal force acting at the vault at a distance x= ½ a from the top the normal force is equal to:

$N_x = H \cos \phi + V_x \sin \phi$

Substituting for x = ½ a the thrust H, the shear force V_x and ϕ=14.036°:

$N_x = 29.9 \times \cos \phi + 7.4 \times \sin \phi = 30.8$ kN

Live load

For the three hinged vault subjected to a live load q_e = 1.0 kN/m² the vertical reaction acting at the supports V_A and V_B follows from: $V_A = V_B = q_e a = 1.0 \times 7.2 = 7.2$ kN
The thrust follows from [6.7]: $H = ½ q_e a^2/f = ½ \times 2.0 \times 7.2^2/1.8 = 14.4$ kN
For x = ½ a the normal force is equal to: $N_{x=a/2} = (H + V_x)^{½} = (14.4^2 + 3.6^2)^{½} = 14.8$ kN

Asymmetric live load

The vault can be subjected to a live load acting asymmetrically. Assume the vault is subjected to load q = 1.0 kN/m acting at the right side. The vertical reaction acting at the support at the left side, V_A, and the right side V_B are respectively:

$V_A = ¼ q_e a$ → $V_A = ¼ \times 1.0 \times 7.2 = 1.8$ kN
$V_B = ¾ q a$ → $V_B = ¾ \times 1.0 \times 7.2 = 5.4$ kN

The thrust H follows from [6.8]: $H = ¼ q_e a^2/f$ → $H = ¼ \times 1.0 \times 7.2^2/1.8 = 7.2$ kN

The asymmetrical load can be considered as a combination of an equally distributed load equal to ½ q and an anti-metrical load equal to ½ q. Due to the anti-metrical load the vault is subjected to a maximum bending moment at maximum for x = ½ a. Substituting q and a into [6.13] gives:

$M_{x\,max} = \pm ^1/_8 \times (½ q a^2)$ → $M_{x\,max} = \pm ^1/_8 \times 0.5 \times 7.2^2 = -3.24$ kNm

For x = ½ a the force V_x is for the unloaded side equal to: $V_x = ¼ q a = ¼ \times 1.0 \times 7.2 = 1.8$ kN ↓

The normal force follows from: $N_x = H \cos \phi + V_x \sin \phi$.
For x = ½ a $\phi = \beta$, the normal force is equal to: $N_x = 7.2 \times \cos \beta + 1.8 \times \sin \beta = 7.4$ kN

Stresses

The stresses acting in the concrete, due to the normal forces and bending moments according to the Theory of Elasticty follow from:

$$\sigma_c = \frac{N E_c}{AE} \pm \frac{M z E_c}{EI}$$

With: z= ½ ×110 = 55 mm; AE = 2.0 × 10⁹ Nmm²; E_c = 2.7 × 10⁴ MPa; EI= 2.8 × 10¹² Nmm²

Table 6.1 shows the forces, bending moments and stresses due to the permanent and asymmetric load for x = ½ a.The stresses due to the normal compressive forces are quite small. For the combination of the permanent and asymmetrical live load the tensile bending stresses are not compensated by the compressive stresses due to the normal forces.

Load	H [kN]	V [kN]	$N_{x=½a}$ [kN]	$\sigma_c = N \times E_c / AE$ [MPa]	M [kNm]	$\sigma_c = M \times z \times E_c / EI$ [MPa]
perm. Load	29.9	15.3	30.8	-0.42	0.25	± 0.13
sym.live load	14.4	7.20	14.8	-0.20	0	0
asym.live load	7.2	5.40	7.4	-0.10	± 3.24	± 1.72

TABLE 6.1 Forces, bending moments and stressses due to the permanent load and the live load acting in the concrete for x = ½ a from the top.

§ 6.4 Two hinged parabolic vault

For a three-hinged curved element the thrust can be calculated with the equations describing the equilibrium of forces and bending moments. A two-hinged vault is statically indeterminate. To calculate the thrust expressions describing the deformations of the arch due to the loads and thrust are also required. Next the deformations of the supports are calculated for an equally distributed load q and a horizontal force H acting on the supports. The structure is schemed as a curved beam supported by two rollers and subjected to an equally distributed load. Due to the symmetry of form and load the top does not move horizontally. To define the horizontal deformation of the roller supports the structure is schematised as clamped at the top. Firstly the deformations of the roller supports are calculated for the half curved element subjected to an equally distributed load and next the deformations of the roller supports are calculated for an horizontal force acting at the roller support. Actually the curved element is supported with two hinges able to resist the thrust. The horizontal deformations of the hinges is zero, then the deformation Δ_H of the force H is equal to the deformation Δ_q of the curved element due to the distributed load. The horizontal force H follows from this expression describing the compatibility: $\Sigma\Delta = 0$. Thus the vault is schematised as a parabolic curved element clamped at the top, subjected to an equally distributed load q and a horizontal force H acting at the roller. The bow of the arch is equal to f, the span of half of the curved element is equal to a, the centre of the coordinates is defined at the top. The parabola is described with the function: $y = f x^2/a^2$.

Deformation due to the distributed load q

Assuming the curved element is subjected by an equally distributed load q. Due to this load the element is subjected to a bending moment: $M_x = ½ q (a^2 - x^2)$

A small part of the curve ds is subjected to a bending moment M_x, Due to this moment this part is subjected to a rotation $d\phi$, with $d\phi = M_x / EI$. The horizontal deformation due to the rotation of the small part is equal to: $\delta_x = M_x (f-y)/EI$

The total deformation Δ_x is equal to the sum of the deformations of every part of the curve. Decreasing the length of the small parts ds will improve the calculation. Thus the horizontal deformation is calculated by integration of the expression for δ_x over the length of the curve between the centre and the support.

$$\Delta_x = \int \frac{M_x (f-y)\, ds}{EI}$$

Substituting the bending moment M_x into this expression results in:

$$\Delta_x = \tfrac{1}{2} q \times \frac{\int (a^2-x^2)(f-y)\,ds}{E I_x}$$

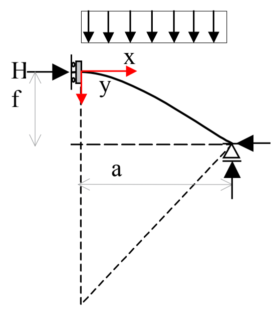

FIGURE 6.12 Half curved parabolic vault subjected to equally distributed load q. aption

The analysis is simplified if the element is constructed with a height increasing from the centre to the supports so $I_x = I_0\, ds/dx$, see [Tus52]. The effect of the increased height is minor if the bow is much smaller than the span or when ½ f/a is small. Now the expression can be formulated as:

$$\Delta_x = \frac{\tfrac{1}{2} q \times \int_{x=0}^{x=a} (a^2-x^2) \times (f - f x^2/a^2)\; ds}{E I_0 \times ds/dx}$$

This expression can be simplified to:

$$\Delta_x = \frac{\tfrac{1}{2} q f \int_{x=0}^{x=a} (x^4/a^2 + a^2 - 2x^2)\, dx}{E I_0}$$

Integration between x = 0 and x = a gives:

$$\Delta_{x=a} = \frac{\tfrac{1}{2} q f\, (x^5/5a^2 + a^2x - 2x^3/3a^2)}{E I_o}\Big|_{x=0}^{x=a} \rightarrow \Delta_{x=a} = \frac{4 q f a^3}{15 E I_o} \qquad [6.27]$$

Deformation due to the horizontal force H

The bending moments acting on the curved element are significantly reduced when the supports are changed from rollers into hinges so the thrust can be resisted. To define the thrust we load the rollers with a horizontal force H acting inward. Due to this force both rollers are moved inward. The deformation due to the force H is calculated as follows. Firstly the bending moments resulting from the force H are calculated:

$$M_x = H f (1 - x^2/a^2)$$

A small part of the curve dx is subjected to a bending moment M_x, Due to this moment this part is subjected to a rotation $d\phi$. With $d\phi = M_x/EI$. The horizontal deformation due to the rotation of the small part is equal to:

$$\delta_x = M_x (f-y)/EI$$

The total deformation is equal to the sum of the deformations of every part of the curve. So the horizontal deformation is calculated by integration of the expression for Δ_x over the length of the curve between the centre and the support.

$$\Delta_x = \frac{\int M_x (f-y)\, ds}{EI}$$

Substituting the bending moment M_x into this expression:

$$\Delta_x = \frac{\int H f (1-x^2/a^2) \times (f-y)\, ds}{E I_x}$$

The analysis is simplified when the element is constructed with a height increasing from the centre to the supports so $I_x = I_o\, ds/dx$. Then the expression can be modified to:

$$\Delta_x = \frac{\int H f\ (1-x^2/a^2) \times (f - f x^2/a^2)\, ds}{E I_0\, ds/dx} \quad \rightarrow \quad \Delta_x = \frac{H f^2 \int (1-x^2/a^2)^2\, dx}{E I_0}$$

This expression can be reduced to:

$$\Delta_x = \frac{H f^2 \int (x^4/a^4 + 1 - 2x^2/a^2)\, dx}{E I_0}$$

Integration between x = 0 and x = a gives:

$$\Delta_{x=a} = \frac{H f^2 (x^5/(5a^4) + x - 2x^3/(3a^2))\big|_{x=0}^{x=a}}{E I_0} \quad \rightarrow \quad \Delta_{x=a} = \frac{8 H f^2 a}{15\ E I_o}$$

[6.28]

Defining the thrust for the equally distributed load

If the support is very stiff and cannot deform then the deformation due to the distributed load must be equal to the deformation of the thrust, $\Sigma\Delta = 0$. Now the thrust H follows from the equations [6.27] and [6.28]:

$$\frac{8 H f^2 a}{15\ E I_o} = \frac{4 q f a^3}{15 E I_o} \quad \rightarrow \quad H = \frac{½ q a^2}{f}$$

Deformation due to a linearly increasing load q

Assuming the curved element is subjected by a load linearly increasing from the centre to the supports: $q_x = q\, x/a$. At the support the load is at maximum and equal to q. Due to this load the curved element is subjected to a bending moment:

$$M_x = {}^1/_6\, q\, a^2 (1- x^3/a^3)$$

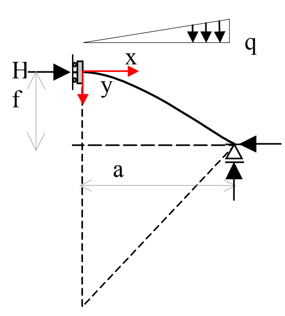

FIGURE 6.13 Half curved parabolic vault subjected to linearly increasing load

A small part of the curve ds is subjected to a bending moment M_x. Due to this moment this part is subjected to a rotation $d\phi$, with $d\phi = M_x / EI$. The horizontal deformation due to the rotation of the small part is equal to:

$$\delta_x = M_x (f-y)/EI$$

The total deformation Δ_x is equal to the sum of the deformations of every part of the curve. Thus the horizontal deformation is calculated by integration of the expression for δ_x over the length of the curve between the centre and the support.

$$\Delta_x = \int \frac{M_x (f-y)\, ds}{EI}$$

Substituting the bending moment M_x into this expression results in:

$$\Delta_x = \frac{q a^2 f \times \int (1 - x^3/a^3) \times (1 - x^2/a^2)\, ds}{6 E I_x}$$

The analysis is simplified if the element is constructed with a height increasing from the centre to the supports so $I_x = I_0\, ds/dx$, see [Tus52]. The effect of the increased height is minor if the bow is much smaller than the span or when f/a is small. Now the expression can be formulated as:

$$\Delta_x = \frac{q a^2 f \times \int (1 - x^2/a^2 - x^3/a^3) + x^5/a^5)\, dx}{6 E I_0}$$

Integration between x = 0 and x = a gives:

$$\Delta_x = \frac{q a^2 f \times [x - {}^1/_3\, x^3/a^2 - {}^1/_4\, x^4/a^3 + {}^1/_6\, x^6/a^5]\,|_{x=0}^{x=a}}{6 E I_o} \rightarrow \Delta_{x=a} = \frac{7 q f a^3}{72 E I_o} \qquad [6.29]$$

Deformation due to the horizontal force H

The bending moments acting on the curved element are significantly reduced when the supports are changed from rollers into hinges so the thrust can be resisted. To define the thrust we load the rollers with a horizontal force H acting inward. Due to this force both rollers are moved inward. The deformation due to the force H follows from [6.28] calculated as follows.

$$\Delta_{x=a} = \frac{8 H f^2 a}{15 E I_o} \qquad [6.28]$$

Defining the thrust for the linearly increasing load

If the support is very stiff and cannot deform then the deformation due to the load must be equal to the deformation of the thrust, $\Sigma\Delta = 0$. Now the thrust H follows from the equations [6.28] and [6.29]:

$$\frac{8 H f^2 a}{15 E I_o} = \frac{7 q f a^3}{72 E I_o} \rightarrow H = \frac{35 q a^2}{192 f} \qquad [6.30]$$

Due to this load the thrust the curved element is subjected to a bending moment:

$$M_x = {}^1/_6\, q a^2 (1 - x^3/a^3) - H (f - y)$$

Substituting H and $y = f x^2/a^2$ into this expression gives:

$$M_x = q\,a^2\,[^1/_6 - {}^1/_6\,x^3/a^3 - {}^{35}/_{192} \times (1 - x^2/a^2)] \quad \rightarrow \quad M_x = q\,a^2\,[{}^{35}/_{192}\,x^2/a^2 - {}^1/_6\,x^3/a^3 - {}^3/_{192}]$$

For x = 0 the bending moment is equal to: $M_{x=0} = -{}^1/_{64}\,q\,a^2 = -0.015625 \times q\,a^2$

The bending moment is at maximum for dM/dx = 0, differentiating M_x gives:

$$dM_x/dx = q\,a^2 \times [+{}^{70}/_{192}\,x/a^2 - ½\,x^2/a^3]$$

The maximum bending moment is found for x/a = 0 and x/a = $^{35}/_{48}$.

For x/a = 0 the bending moment is equal to: $M_{x=0} = -\ 0.015625 \times q\,a^2$

For x/a = $^{35}/_{48}$ the bending moment is equal to: $M_{max} = +\,0.016682 \times q\,a^2$

For the three hinged vault the maximum bending moment is found for x = = $^2/_3$ a:

$$M_{max} = {}^2/_{81}\,q\,a^2 = 0.0246914 \times q.a^2 \qquad [6.16]$$

For the two-hinged vault the bending moments are smaller than for the three-hinged vault. In practice the supports will deform slightly, for example due to the lengthening of the tie. Due to the deformation of the tie the thrust will decrease, so the bending moment at the top will decrease and the bending moment for x > 0 will increase.

Conclusions

For the parabolic two hinged vault subjected to an equally distributed load the thrust is equal to the thrust calculated for a three hinged vault, provided the supports do not deform. For the three hinged vault subjected to an equally distributed load the bending moments are equal to zero, so for the equally distributed load the two hinged vault is also subjected to normal forces only.

For the parabolic two hinged vault subjected to a linearly increasing load the thrust is slightly larger than the thrust for a three hinged vault, provided the supports do not deform. The bending moments are smaller than the maximum bending moment calculated for the three-hinged vault, subjected to a linearly increasing load.

In practise the tie, connecting both supports, will lengthen, generally the lengthening of the tie will be quite small, but nevertheless due to this deformation the thrust will decrease. For a two-hinged vault, subjected to an equally distributed load, the structure will be subjected to bending due to the decrease of the thrust. For a two-hinged vault, subjected to a linearly increasing load, the bending moment at the top will decrease and the bending maximum moment for x > 0 will increase, due to the lengthening of the tie and the reduction of the thrust. Often for statically indeterminate arches and vaults the thrust can be calculated easily, if the supports cannot deform much, by considering the vault as a statically determinate vault with a fictive hinge at the top.

§ 6.5 Parabolic vault, strengthened with a simple truss composed of two diagonals

Strengthening a vault with two diagonals ties running from the supports to the top increases the resistance much. Thanks to these diagonals the bending moments are reduced substantially. To decrease the bending moments due to the asymmetrical loads the vault is strengthened with the ties running diagonally from the crown to the supports. The span and rise of the arch is equal to respectively 2 a and f. The diagonals are running from the top to the supports. The inclination of these diagonals is equal to β, with tan β = f/a. The strengthened vault is statically determinate, the distribution of the loads is effected by the stiffness of the vault and the truss. To analyse the transfer of the loads the deformations of the truss and vault will be defined for a concentrated load acting at the top, an equally distributed load and an anti-metrical load. The structure is assumed to be prefabricated and composed of two parts connected with a hinge at the top. The structure is schemed as a three hinged vault resting on two simple supports. To simplify the analysis the elongation of the tie is neglected.

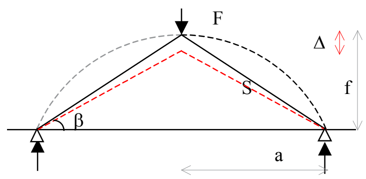

FIGURE 6.14 Deformation of the trussed frame composed of two diagonals due to a concentrated vertical force F acting at the top

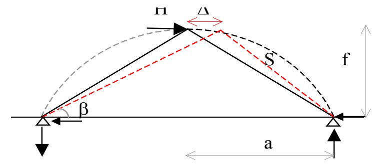

FIGURE 6.15 Deformation of the trussed frame composed of two diagonals due to a concentrated horizontal force H acting at the top

The deformation of a simple truss composed of two diagonals due to concentrated loads

The deformation of the truss, subjected to a vertical load F, acting at the top, follows from:

$$\Delta_{TF} = \frac{F\,a}{2\,A_T E_T \cos\beta \sin^2\beta} \qquad [6.31]$$

The deformation of the truss, subjected to a horizontal load H, acting at the top, follows from:

$$\Delta_{TH} = \frac{H\,a}{2\,A_T E_T \cos^3\beta} \qquad [6.32]$$

The deformation of the vault

The center of the coordinates of the parabolic three hinged vault is positioned at the top, see figure 6.2. The deformation of the vault at a chosen point is defined according to the Theory of Maxwell-Mohr with:

$$\Delta = \int \frac{M'\,M\,ds}{EI} + \int \frac{N'\,N\,ds}{EA} \qquad [6.33]$$

With:
M' is the bending moment due to a force F = 1 acting at the chosen point parallel to the deformation;
M = the bending moment due to the load;
N' = the normal force due to a force F = 1 acting at the chosen point parallel to the deformation;
N = the normal force due to the load.

The deformation of the vault subjected by vertical concentrated force acting at the top

The vault is subjected to a concentrated force F acting at the top. Due to this load the thrust is equal to: H =½ F a/f. The bending moment follows from:

$$M_x = ½\,F\,x - H\,y = ½\,F\,a\,(x^2/a^2 - x/a) \qquad [6.22]$$

For the parabolic vault the normal force is equal to:

$$N_x = H\cos\phi + V_x\sin\phi \quad \rightarrow \quad N_x = H\cos\phi\,(1 + V_x\tan\phi/ H)$$

Substituting $\tan\phi = y' = 2fx/a^2$ and $V_x = ½ F$ and $H = ½ F a/f$ into this expression gives:

$$N_x = ½ F (a/f) \cos\phi\,(1 + 2f^2x/a^3) \qquad [6.34]$$

For low rise vaults $\cos\phi$ is approximately equal to $\cos\beta$, substituting $\cos\phi = \cos\beta$ gives:

$$N_x = ½ F (a/f) \times \cos\beta \times (1 + 2f^2x/a^3)$$

Substituting M'_x, M_x, N'_x and N_x into expression [6.33] gives:

$$\Delta_F = \frac{F a^2}{2\,EI} \int_0^a [x^2/a^2 - x/a]^2\,ds + \frac{F a^2 \cos^2\beta}{2\,AE f^2} \int_0^a \left[1 + \frac{2f^2x}{a^3}\right]^2 ds$$

For low rise vaults the length ds is approximately equal to $dx/\cos\beta$. Integrating this expression between the boundaries $x = 0$ and $x = a$ gives:

$$\Delta_F = \frac{F a^3}{60\,EI\cos\beta} + \frac{F a \cos\beta}{2\,AE}\left[\frac{a^2}{f^2} + \frac{4f^2}{3a^2} + 2\right] \qquad [6.35]$$

For $f = ¼\,a$ the deformation is equal to:

$$\Delta_F = \frac{F a^3}{60\,EI\cos\beta} + \frac{217\,F a \cos\beta}{24\,AE f^2} \qquad [6.35']$$

Assume the truss is subjected to a vertical force αF and the vault is subjected to a force $(1 - \alpha) F$. At the top the displacement of the vault is equal to the displacement of the truss, the factor a follows from:

$$\Delta_F(1 - \alpha) = \alpha\,\Delta_{TF} \quad \rightarrow \quad \alpha = \Delta_F/(\Delta_F + \Delta_{TF}) \qquad [6.36]$$

Example

For the vault described previously the deformation is defined if the structure is strengthened with two diagonals and subjected to a vertical force $F = 1.0$ kN. Again the span and rise are equal to $l = 2a = 14.4$ m and $f = 1.8$ m. The tangent of the angle between the diagonals and tie is equal to $\tan\beta = 0.25$, then $\cos\beta = 0.97$. The stiffness of the vault is equal to: $AE = 2.0 \times 10^9$ N and $EI = 2.8 \times 10^{12}$ Nmm^2. The area and Young's modulus of the diagonals Ø100-4 are respectively $A_T = 1206$ mm^2 and 2×10^5 MPa.

The deformation of the truss due to a force $F = 1.0$ kN is according to [6.31] equal to:

$$\Delta_{TF} = \frac{1.0 \times 10^3 \times 7200}{2 \times 1206 \times 2.0 \times 10^5 \times \cos\beta \sin^2\beta} = 0.26 \text{ mm}$$

The deformation of the vault due to the concentrated load equal to $F = 1.0$ kN is according to [6.35'] equal to:

$$\Delta_F = \frac{1.0 \times 10^3 \times 7200^3}{60 \times 2.8 \times 10^{12}\,'\,\cos\beta} + \frac{217 \times 1.0 \times 10^3 \times 7200 \times \cos\beta}{24 \times 2.0 \times 10^9} = 2.29 + 0.03 \text{ mm}$$

Substiuting the deformation of the truss Δ_{TF} and vault Δ_F into [6.36]: $\alpha = \dfrac{2.29 + 0.03}{2.29 + 0.03 + 0.26} = 0.9$

The force acting at the truss is equal to $\alpha F = 0.9$ kN, thus the better part of the concentrated load F acting at the top of the vault is transferred by the truss. The diagonals are subjected to a force:

$$S = ½\,\alpha F/\sin\beta = 1.86 \text{ kN}.$$

The deformation of the vault subjected by a horizontal concentrated force acting at the top

The vertical reaction forces acting at the supports is equal to: $H f - V_A a = 0 \rightarrow V_a = ½ H f/a$

The bending moment M_x at a distance x from the support at the left side follows from [6.24]:

$$M_x = ½ H f (x^2/a^2 - x/a) \qquad [6.24]$$

Thus for a concentrated force H = 1 acting at the top the bending moment is equal to:

$$M_x = ½ f (x^2/a^2 - x/a) \qquad [6.24']$$

For a vault the normal force is equal to:

$$N_x = ½ H \cos\phi + V_x \sin\phi \quad \rightarrow \quad N_x = ½ H \cos\phi\,(1 + V_x \tan\phi/H)$$

For a low rise vault the angle f is very small, so $\cos\phi \approx \cos\beta$. Substituting $\tan\phi = y' = 2 f x/a^2$ and $V_a = ½ H \times f/a$ into this expression gives:

$$N_x = ½ H \cos\beta\,(1 + 2 f^2 x/a^3) \qquad [6.37]$$

Thus for a concentrated force H = 1 acting at the top the normal force is equal to:

$$N_x = ½ \cos\beta\,(1 + 2 f^2 x/a^3)$$

Substituting M'_f, M_f, N'_f and N_f into [6.31] gives:

$$\Delta_H = \frac{2 \times ½ \times ½ H f^2}{EI} \times \int_0^a [x^2/a^2 - x/a]^2 ds + \frac{2 \times ½ \times ½ H \cos^2\beta}{AE} \times \int_0^a \left[1 + \frac{2 f^2 x}{a^3}\right]^2 ds$$

For a low rise vault the length ds is approximately equal to dx/cos β. Integrating this expression gives:

$$\Delta_H = \frac{H f^2 a}{60\, EI \cos\beta} + \frac{H a \cos\beta\,(1 + {}^4/_3 f^4/a^4 + 2 f^2/a^2)}{2\, EA} \qquad [6.38]$$

For f = ¼ a the deformation is equal to: $$\Delta_H = \frac{H a^3}{960 \times EI \cos\beta} + \frac{217 \times H \cos\beta}{384 \times AE} \qquad [6.38']$$

Assume the truss is subjected to a vertical force α H and the vault is subjected to a force (1 - α) H. At the top the displacement of the vault is equal to the displacement of the truss, the factor α follows from:

$$\Delta_H\,(1 - \alpha) = \alpha\,\Delta_{TH} \quad \rightarrow \quad \alpha = \Delta_H/(\Delta_H + \Delta_{TH}) \qquad [6.39]$$

Example

For the strengthened vault described previously the deformation is defined for the vault, subjected to a horizontal force H = 1.0 kN.
The deformation of the truss due to a force H = 1.0 kN is according to [6.32] equal to:

$$\Delta_{TH} = \frac{½ \times 10^3 \times 7200}{206 \times 2 \times 10^5 \times \cos^3\beta} = 0.016 \text{ mm}$$

The deformation of the vault due to the concentrated force H = 1.0 kN is equal to:

$$\Delta_H = \frac{10^3 \times 7.2^3 \times 10^9}{960 \times 2.8 \times 10^{12} \times \cos\beta} + \frac{217 \times 10^3 \times 7200 \times \cos\beta}{384 \times 2.0 \times 10^9} = 0.143 + 0.002 \text{ mm}$$

Substiute the deformation of the truss and vault into [6.39]:

$$\alpha = \frac{0.143 + 0.002}{0.143 + 0.002 + 0.016} = 0.9$$

Thus the force acting at the truss is equal to α.H = 0.9 kN. The truss transfers the better part of the concentrated horizontal force. The force acting at the diagonals is equal to S = ½ αH/cos β = 0.5 kN.

Equally distributed load

The vault is subjected to an equally distributed load q. The vertical reactions acting at the supports at the left and right side, V_A and V_B, are equal to: $V_A = V_B$ = q a. The thrust H, acting at the supports follows from [6.7]: $H = ½ q a^2/f$

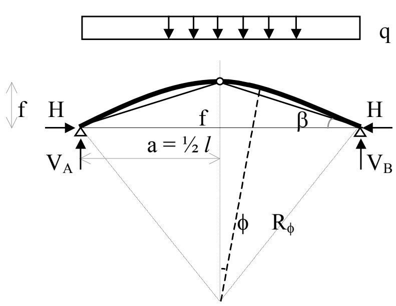

FIGURE 6.16 Vault strengthened with two diagonals, subjected to an equally distributed load

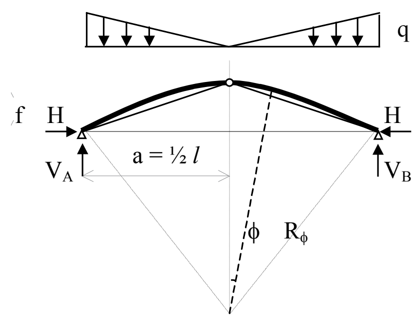

FIGURE 6.17 Vault strengthened with two diagonals, subjected to an increasing load

For an equally distributed load the bending moments are equal to zero. This vault is subjected to normal forces N_x only. The normal forces are calculated with:

$$N_x = H \cos\phi + V_x \sin\phi \qquad \rightarrow \qquad N_x = H \cos\phi\,(1 + q\,x \tan\phi\,)$$

Substituting H into this expression gives for a low rise vault:

$$N_x = ½\, q\, a^2/f \times \cos\beta\ [1 + 4\,(f/a)^2 \times (x/a)^2]$$

For a vault subjected to a concentrated load F the normal force follows from [6.34]:

$$N_x = ½\, F\, a/f \times \cos\beta\ (1 + 2\, f^2\, x/a^3)$$

The deformation of the vault at the top is defined with the Theory of Maxwell/Mohr with [6.33]:

$$\Delta = \int \frac{M'\,M\,ds}{EI} + \int \frac{N'\,N\,ds}{EA} \qquad [6.33]$$

The length of a small piece of the parabola follows from: ds = dx/cos ϕ Substituting N_x and ds into [6.33] gives:

$$\Delta_q = \frac{2\ q.a^3 \cos\beta}{4\,EA\,f^2} \times \int_0^a [1 + 4\,f^2 x^2/a^4\,] \times [1 + 2\,f^2\,x/a^3)^2]\ dx$$

Integrating this expression gives:

$$\Delta_q = \frac{q\,a^2 \cos\beta}{2\,EA} \times \left(\frac{a^2}{f^2} + \frac{7}{3} + \frac{2\,f^2}{3\,a^2} \right) \qquad [6.40]$$

For f = ¼ a: $\Delta_q = \frac{443 \times q\, a^2 \cos\beta}{48\, EA} \rightarrow \Delta_q = \frac{9.229 \times q\, a^2 \cos\beta}{EA}$ [6.40']

Assume the truss is subjected to a vertical force α F and the vault is subjected to a force α F. At the top the displacement of the vault is equal to the displacement of the truss. The factor a follows from:

$$\Delta_q - \alpha\,\Delta_F = \alpha\,\Delta_{TF} \rightarrow \alpha = \Delta_q/(\Delta_F + \Delta_{TF}) \quad [6.41]$$

Example

For the vault, described previously the deformation is defined for q = 1.0 kN/m.Due to a vertical force F = 1.0 kN the deformation of the truss is equal to:

$$\Delta_{TF} = \frac{10^3 \times 7200}{1206 \times 2 \times 10^5 \times \sin^2\beta \cos\beta} = 0.26 \text{ mm}$$

Due to the equally distributed load q = 1.0 kN/m the deformation of the vault is according to [6.40'] equal to:

$$\Delta_q = \frac{9.229 \times 7200^2 \cos\beta}{2.0 \times 10^9} = 0.23 \text{ mm}$$

The deformation of the vault due to the concentrated force F = 1.0 kN is according to [6.35'] equal to:

$$\Delta_F = \frac{1.0 \times 10^3 \times 7200^3}{60 \times 2.8 \times 10^{12} \cos\beta} + \frac{217 \times 1.0 \times 10^3 \times 7200 \times \cos\beta}{24 \times 2.0 \times 10^9} = 2.29 + 0.03 \text{ mm}$$

Assume the truss is subjected to a vertical force α F and the vault is subjected to a force α F. The factor α follows from [6.41]: $\alpha = \Delta_q/(\Delta_F + \Delta_{TF})$

Substituting Δ_q = 0.23 mm, Δ_F = 2.29 + 0.03 mm and Δ_{TF} = 0.26 mm:

$$\alpha = \frac{0.23}{2.29 + 0.03 + 0.26} = 0.9$$

Thus the force acting at the truss is equal to **α F = 0.9 kN** and the force acting at the diagonals is equal to **S = ½ α F/sin β = 0.5 kN**. The better part of the equally distributed load is transferred by the vault.

Linearly increasing load

A load is linearly increasing from the top to the supports. At the supports the maximum load is equal to q. At a distance x from the top the load is equal to $q_x = q\,x/a$. For this load the deformation at the top will be defined.

The vertical reaction acting at the support A and B is equal to: $V_A = V_B = ½\, c\, q\, a$.

The thrust H acting at the supports follows from [6.14]: $H = {}^1/_6\, c\, q\, a^2/f$

The bending moment M_x at a distance x from the top follows from [6.15]:

$$M_x = {}^1/_6\, c\, q\, a^2 (x^2/a^2 - x^3/a^3) \quad [6.15']$$

The normal force acting at the vault is equal to: $N_x = H \cos\phi\, (1 + V_x \tan\phi)$

Substituting $H = {}^1/_6\, q\, a^2/f$ and $V_x = ½\, q\, x^2/a$ gives with cos ϕ ≈ cos β :

$$N_x = {}^1/_6\, c\, q\, a^2/f \times \cos\beta\, [\,1 + 6\, x^3 f^2/a^5\,] \quad [6.42]$$

The deformation of the vault at the top is defined according to the Theory of Maxwell-Mohr with:

$$\Delta = \int \frac{M' M\, ds}{EI} + \int \frac{N' N\, ds}{EA} \qquad [6.33]$$

For a concentrated vertical force F = 1 acting at the top the bending moment follows from [6.22]:

$$M_x = ½\, a \times (x^2/a^2 - x/a) \qquad [6.22']$$

For a vault subjected to a concentrated load F = 1 the normal force follows from [6.34']:

$$N_x = ½\, a/f \times (1 + 2\, f^2 x/a^3) \cos\beta \qquad [6.34']$$

Substituting M'_f, M_f, N'_f and N_f into [6.33] gives with ds ≈ dx/cos β:

$$\Delta_{cq} = \frac{c\, q\, a^3}{6\, EI \cos\beta} \int_0^a [x^2/a^2 - x/a] \times [x^2/a^2 - x^3/a^3]\, dx + \frac{q\, a^3 \cos\beta}{6\, AE\, f^2} \times \int_0^a [1 + \frac{2\, f^2 x}{a^3}] \times [1 + \frac{6\, f^2 x^3}{a^5}]\, dx$$

Integrating this expression gives:

$$\Delta_{cq} = -\frac{c\, q\, a^4}{360\, EI \cos\beta} + \frac{c\, q\, a^4 \cos\beta}{6\, AE\, f^2} \times [1 + \frac{5\, f^2}{2\, a^2} + \frac{12\, f^4}{5\, a^4}] \qquad [6.43]$$

For f = ¼ a the deformation is equal to: $\Delta_{cq} = -\frac{c\, q\, a^4}{360\, EI \cos\beta} + \frac{373\; c\, q\, a^2 \cos\beta}{120\, AE}$ [6.43']

Assuming the truss is subjected to a vertical force α F acting downward and the vault is subjected to a force α F acting upward. At the top the displacement of the vault is equal to the displacement of the truss. The factor α follows from:

$$\Delta_q - \alpha\, \Delta_F = \alpha\, \Delta_{TF} \quad \rightarrow \quad \alpha = \Delta_q/(\Delta_F + \Delta_{TF}) \qquad [6.41]$$

Example

For the vault, described previously the deformation is defined for a linearly increasing load $q_x = c\, q\, x/a$ with q = 1.0 kN/m.

Due to a vertical force F = 1.0 kN the deformation of the truss is equal to:

$$\Delta_{TF} = \frac{10^3 \times 7200}{1206 \times 2 \times 10^5 \times \sin^2\beta \cos\beta} = 0.26 \text{ mm}$$

Due to the linear increasing load $q_x = q\, x/a$ with c q = 1.0 kN/m the deformation of the vault is according to (6.43') equal to:

$$\Delta_{c.q} = -\frac{1.0 \times 7200^4}{360 \times 2.8 \times 10^{12} \cos\beta} + \frac{373 \times 1.0 \times 7200^2 \cos\beta}{120 \times 2.0 \times 10^9} = -2.75 + 0.08$$

The deformation of the vault due to the concentrated force F = 1.0 kN is according to [6.35'] equal to:

$$\Delta_F = \frac{1.0 \times 10^3 \times 7200^3}{60 \times 2.8 \times 10^{12}} + \frac{217 \times 1.0 \times 10^3 \times 7200}{24 \times 2.0 \times 10^9} = 2.29 + 0.03 \text{ mm}$$

Assume the truss is subjected to a vertical force α F acting downward and the vault is subjected to a force α F acting upward. The factor α follows from [6.41] : $\alpha = \Delta_q/(\Delta_F + \Delta_{TF})$

Substituting Δ_q = 0.227 mm, Δ_F = 2.29 + 0.03 mm and Δ_{TF} = 0.26 mm gives:

$$\alpha = \frac{-2.75 + 0.08}{2.29 + 0.03 + 0.26} = -1.0$$

Thus the force acting at the truss is equal to α F = -1.0 kN, the negative sign shows the force α F is acting downward ↓. The force acting at the diagonals is equal to S = ½ α F/sin β = -2.06 kN. The better part of the linearly increasing load is transfered by the vault.

Deformation of the vault due to an anti-metrical load

For a vault subjected to an anti-metrical load q the vertical reaction acting at the support at the left and right side are respectively equal to: V_A = -½ q × a ↓ and V_B = ½ q × a↑. The thrust is equal to zero.

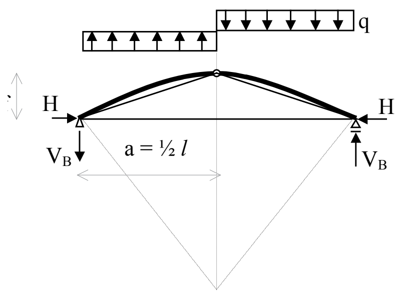

FIGURE 6.18 Vault subjected to an antimetrical load

For the right part the bending moment M_x follows from [6.12]: M_x = ½ q a² (x/a - x²/a²)

The normal force follows from: $N_x = V_x \sin \phi$.

For a low rise vault the angle ϕ is quite small so cos $\phi \approx$ cos β. Substituting V_x = q x - ½ q a and tan ϕ = 2 f x/a² gives:

$$N_x = q\, f\, (2\, x^2/a^2 - x/a) \cos \beta \qquad [6.44]$$

Due to an anti-metrical load the vault will deform horizontally to the left side. The deformation of the vault at the top is defined according to the Theory of Maxwell-Mohr with (6.33):

$$\Delta = \int \frac{M'\, M\, ds}{EI} + \int \frac{N'\, N\, ds}{EA} \qquad [6.33]$$

For a concentrated force H = -1 acting at the top the bending moment and normal force are respectively equal to: $M_x = -½\, f\, (x^2/a^2 - x/a)$ and $N_x = -½\, (1 + 2\, f^2 x/a^3) \cos \beta$

Substituting M'_f, M_f, N'_f and N_f into (6.32) gives with ds ≈ dx/cos β:

$$\Delta_q = \frac{2\, q\, f\, a^2}{4\, EI \cos \beta} \times \int_0^a [x^2/a^2 - x/a]^2\, dx + \frac{q\, f \cos \beta}{AE} \times \int_0^a [2\, x^2/a^2 - x/a] \times [-1 - 2\, f^2\, x/a^3]\, dx$$

Integrating this expression gives:

$$\Delta_q = \frac{q\, f\, a^3}{60 \times EI \cos \beta} - \frac{q\, a\, f \cos \beta}{6\, AE} \times \left[1 + \frac{2\, f^2}{a^2}\right] \qquad [6.45]$$

For f/a = ¼ the deformation is equal to:

$$\Delta_q = \frac{q\, a^4}{240 \times EI \cos \beta} - \frac{3 \times q\, a^2 \cos \beta}{64 \times AE} \qquad [6.45']$$

Due to a horizontal force H the deformation of the vault follows from [6.38]:

$$\Delta_H = \frac{H\, f^2\, a}{60\, EI \cos \beta} + \frac{H\, a \cos \beta\, (1 + {}^4/_3 f^4/a^4 + 2\, f^2/a^2)}{2\, EA} \qquad [6.38]$$

For f/a = ¼ the deformation is equal to: $$\Delta_H = \frac{H\,a^3}{960\ EI\cos\beta} + \frac{217\ H\cos\beta}{384\ AE}$$ [6.38′]

The deformation of the truss, subjected to a horizontal load H acting at the top, follows from [6.32]:

$$\Delta_{TH} = \frac{H\,a}{2\,AE\cos^3\beta}$$ [6.32]

Assume the truss is subjected to a force α H acting to the left side and the vault is subjected to a force α H acting to the right side. At the top the displacement of the vault is equal to the displacement of the truss, the factor a follows from:

$$\Delta_q - \alpha\,\Delta_H = \alpha\,\Delta_{TH} \quad \rightarrow \quad \alpha = \Delta_q/(\Delta_H + \Delta_{TH})$$ [6.46]

Example

For the vault, described previously, the deformation is defined for an antimetrical load q = 1.0 kN/m. The deformation of the vault due to the anti-metrical load q = 1.0 kN/m with f = ¼ a follows from (6.45′):

$$\Delta_q = \frac{1.0 \times 7200^4}{240 \times 2.8 \times 10^{12}\cos\beta} - \frac{3 \times 1.0 \times 7200^2\cos\beta}{64 \times 2.0 \times 10^9} = 4.12 - 0.001$$

The deformation of the truss due to a force H = 1.0 kN is according to (6.32) equal to:

$$\Delta_{TH} = \frac{½ \times 10^3 \times 7200}{1206 \times 2 \times 10^5 \times \cos^3\beta} = 0.016\text{ mm}$$

The deformation of the vault due to the concentrated force H = 1.0 kN is equal to:

$$\Delta_H = \frac{10^3 \times 7.2^3 \times 10^9}{960 \times 2.8 \times 10^{12}\cos\beta} + \frac{217 \times 10^3 \times 7200 \times \cos\beta}{384 \times 2.0 \times 10^9} = 0.142 + 0.002\text{ mm}$$

Substituting the deformation of the truss and vault into [6.46] gives:

$$\alpha = \frac{4.12 - 0.001}{0.142 + 0.002 + 0.016} = 25.5$$

Due to the anti-metrical load q = 1.0 kN/m the truss is subjected to a force α H = 25.5 kN acting to the left. The diagonals are subjected to a force S = ± ½ α H/cos β = ± 13.1 kN.

§ 6.6 Example three hinged vault, strengthened with truss composed of two diagonals.

To show the effect of the strengthening the three-hinged vault described in paragraph 6.3 is strengthened with two diagonals running from the supports to the top. The span is equal to l = 2 a = 14.4 m. The rise is of the swallow vault is equal to f = l/8 = 1.8 m. The structure is composed of concrete fusées and reinforced with steel. In a section with a width of 1.0 m a number of eleven fusées are placed with a spacing of 10 mm. The centre-to-centre distance of the ceramic elements is equal to 90 mm. The thickness of the vault is 110 mm. To resist the thrust, steel bars Ø25 are made at a centre-to-centre distance of 1.0 m. The vault is reinforced with rebars Ø8 – 180 at the top and bottom with a covering of 15 mm at the top and bottom of the sections of the vault. Distribution bars are not used.

For a part of the roof with width of 1.0 m the areas and second moment of the area of the concrete, fusées and steel are calculated before.According to the calculations made for a factory in Dongen, the Young's modulus of the fusées, concrete C12/15 and reinforcement is respectively equal to E_f = 1.7 × 10^4 MPa, E_c = 2.7 × 10^4 MPa and E_s = 2.0 × 10^5 MPa The distribution and transfer of the loads is defined according to the Theory of Elasticity. The stiffness is calculated by multiplying the area and second moment of the area with the Young's modulus. To simplify the calculation a ratio n_f and n_s is introduced with: $n_f = E_f / E_s = 0.63$ and $n_s = E_s/E_c = 7.4$. Thus EA and EI are calculated with respectively:

$$EA = E_c (A_c + n_f A_f + n_s A_s) \quad [6.25]$$

$$EI = E_c (I_c + n_f I_f + n_s I_s) \quad [6.26]$$

Substituting the values for E_c, I_c, I_f, I_s, A_c, A_f, A_s, n_f and n_s, into the equations [6.25] and [6.26] gives for this vault:

$$EA = 2.7 \times 10^4 \times (54.71 \times 10^3 + 0.63 \times 24.19 \times 10^3 + 7.4 \times 2 \times 279) = 2.0 \times 10^9 \text{ N}$$

$$EI = 2.7 \times 10^4 \times (88.80 \times 10^6 + 0.63 \times 15.12 \times 10^6 + 7.4 \times 0.72 \times 10^6) = 2.8 \times 10^{12} \text{ Nmm}^2$$

This roof is designed for a live load equal to p_e = 1.0 kN/m² and a permanent load equal to q_g = 2.0 kN/m².

Permanent load

Due to the dead load the vault with a width of 1.0 m is subjected to a surface load equal to: q_g = 2.0 kN/m. At the supports for x = a the maximum load is equal to: $q_{max} = q\,(1+ 4 f^2/a^2)^{½}$

To simplify the calculations for low-rise vaults the dead load is assumed to be increasing linearly from q at the top to (1+c).q at the support. For these low-rise vaults the parameter c can be found by assuming for the linearly increasing load the maximum load at the support equal to the maximum load for the dead load, thus:

$$q_{max} = q\,(1+ 4 f^2/a^2)^{½} = q\,(1 + c) \quad \rightarrow \quad c = (1+ 4 f^2/a^2)^{½} - 1$$

For f = ¼ a the maximum load at the support is equal to q_{max} = 1.118 q, thus the parameter c is equal to 0.118.

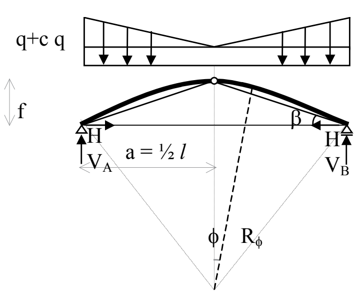

FIGURE 6.19 Parabolic vault subjected to an increasing load q (1+ c x/a)

The vertical reaction V acting at the support A or B is equal to:

$$V = q\,a + ½\,c\,q\,a = 2.0 \times 7.2 \times (1.0 + 0.118/2) = 15.25 \text{ kN}$$

The thrust H follows from expression [6.17]: $H = (1 + {}^{1}/_{3}\, c) \times ½\, q\, a^2/f = 29.93$ kN

For an equally distributed load q = 1.0 kN/m the force α F acting at the truss is equal to 0.09 kN. Thus for an equally distributed load q = 2.0 kN/m the force is equal to α F = 0.18 kN. The compressive force S acting at the diagonals is equal to: $S = ½\, \alpha\, F/\sin\beta = -0.37$ kN.

For a linearly increasing load c q = 1.0 kN/m the force F acting at the truss is equal to −1.0 kN. Thus for an equally distributed load c q = 0.118 × 2.0 kN/m the force is equal to F = -0.24 kN. The force S, tensioning the diagonals, is equal to: $S = ½\, \alpha\, F/\sin\beta = +0.49$ kN.

For a concentrated vertical force F acting at the top the bending moment follows from [6.22]:

$$M_x = ½\, F\, a\, (x^2/a^2 - x/a) \qquad [6.22]$$

For x = ½ a the bending moment due to the concentrated force is equal to $M = -{}^{1}/_{8}\, F\, a$. Substituting the concentrated force α F = 0.18 - 0.24 = -0.06 kN gives:

$$M_{x=a/2} = -0.06 \times 7.2/8 = -0.05 \text{ kNm}$$

For the linearly increasing load c q the bending moment for the not-strengthened vault follows from (6.15):

$$M_x = {}^{1}/_{6}\, c\, q\, (x^2 - x^3/a) \qquad [6.15]$$

For x = ½ a this bending moment is equal to: $M_{x=a/2} = {}^{1}/_{48}\, c\, q\, a^2$.

Substituting for the increasing permanent load c = 0.118 and q = 2.0 kN/m gives:

$$M_{x=a/2} = {}^{1}/_{48} \times 0.118 \times 2.0 \times 7.2^2 = +0.25 \text{ kNm}$$

Thus the resulting bending moment due to the load and the upward force F is equal to:

$$M_{x=a/2} = +0.25 - 0.06 = +0.19 \text{ kNm}$$

At a distance x from the top the normal force is equal to: $N_x = H \cos\phi \times (1 + V_x \tan\phi)$

Substituting for x = ½ a the thrust H, V_x and $\tan\beta = f/a$, $\beta = 14.036°$ gives:

$$N_{x=a/2} = 29.93 \times \cos\beta + [2.0 \times 3.6 \times (1 + 0.118/4)] \times \sin\beta - 0.06/(2 \sin\beta) = 30.7 \text{ kN}$$

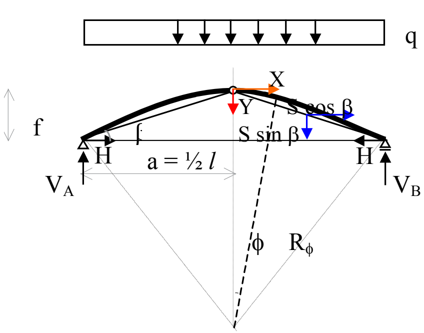

FIGURE 6.20 Parabolic vault subjected to an equally distributed live load

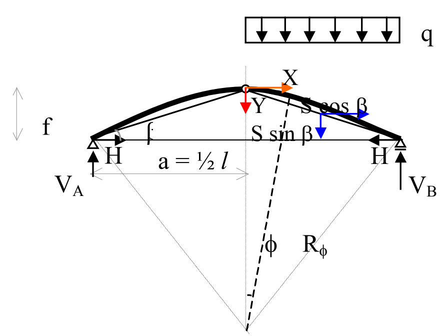

FIGURE 6.21 Parabolic vault subjected to an asymmetric liveload q.

Live load

For the three hinged vault subjected to a live load q_e = 1.0 kN/m² the vertical reaction acting at the supports V_A and V_B follows from: $V_A = V_B = q_e a = 1.0 \times 7.2 = 7.2$ kN
The thrust follows from (6.7): $H = ½ q_e a^2/f = ½ \times 2.0 \times 7.2^2/1.8 = 14.4$ kN

For an equally distributed load q = 1.0 kN/m the force F acting at the truss is equal to 0.09 kN. The compressive force S acting at the diagonals is equal to: **S = ½ F/sin β = 0.19 kN.**

For a concentrated vertical force F acting at the top the bending moment follows from [6.22]:

$$M_x = ½ F a (x^2/a^2 - x/a) \quad [6.22]$$

For x = ½ a, the bending moment is equal to $M_{x=a/2} = -{}^1/_8$ F.a. Substituting α F = 0.09 kN gives:

$$M_{x=a/2} = -0.09 \times 7.2/8 = -0.08 \text{ kNm}$$

The normal forces acting at the vault are reduced slightly due to the upward force α F = 0.09 kN acting at the top of the vault. For x = ½ a the normal force is equal to:

$$N_{x=a/2} = -(H + V_x)^{½} + ½\, \alpha\, F/\sin\beta$$

$$N_{x=a/2} = -(14.4^2 + 3.6^2)^{½} + ½ \times 0.09/\sin\beta = 14.84 - 0.19 = 14.65 \text{ kN}$$

Asymmetric live load

The vault can be subjected to a live load acting asymmetrically. The assumption is made that the vault is subjected to a load q = 1.0 kN/m acting at the right side. The vertical reaction acting at the support at the left side, V_A, and the right side V_B are respectively:

$V_A = ¼\, q\, a \rightarrow V_A = ¼ \times 1.0 \times 7.2 = 1.8$ kN
$V_B = ¾\, q\, a \rightarrow V_B = ¾ \times 1.0 \times 7.2 = 5.4$ kN
The thrust H follows from [6.8]: $H = ¼\, q\, a^2/f \rightarrow H = ¼ \times 1.0 \times 7.2^2/1.8 = 7.2$ kN

The asymmetrical load can be considered as composed of an equally distributed load equal to ½ q and an anti-metrical load equal to ½ q. Thus this asymmetrical load can be considered as the combination of a symmetrical load q = ½ × 1.0 = 0.5 kN/m and an anti-metrical load q = ½ × 1.0 = 0.5 kN/m. Due to the antimetrical load the vault is subjected to bending moments. For x = ½ a the bending moment is at maximum:

$$M_{x=a/2} = \pm {}^1/_8 \times (½\, q) \times a^2 \rightarrow M_{x=a/2} = \pm {}^1/_8 \times 0.5 \times 7.2^2 = -3.24 \text{ kNm}$$

For an equally distributed load q = 1.0 kN/m the force F acting at the truss is equal to 0.09 kN. Thus for q = 0.5 kN/m the force α F acting at the truss is equal to 0.045 kN. For x = ½ a the bending moment due to this force α F acting at the top of the vault is equal to M = ${}^1/_8$ α F a. Substituting α F = 0.045 gives:

$$M_{x=a/2} = -0.045 \times 7.2/8 = -0.04 \text{ kNm}$$

Due to the asymmetrical load the vault will deform sideward. The truss will decrease the horizontal deformation. The asymmetrical load can be considered as the combination of a symmetrical load q = ½ × 1.0 kN/m and an anti-metrical load q = ½ × 1.0 kN/m.

Due to the anti-metrical load q = 1.0 kN/m the truss is subjected to a force **α.H = 25.5 kN**. Thus for an anti-metrical load equal to q = 0.5 kN/m the truss is subjected to a force: **α H = 12.75 kN**.

The force S acting at the diagonals is equal to: $S = ½\,\alpha\, H/\cos\beta = \pm 6.6$ kN.

Due to the force S acting at the diagonal at the right side the vault is subjected to a bending moment:

$$M_x = (S\cos\beta)\, y - (S\sin\beta)\, x$$

For x = ½ a the bending moment due to the force S is equal to: $M_x = S\cos\beta \times ¼ f - S\sin\beta \times ½ a$

Substituting S = 6.6 kN and f = 1.8 m gives: $M_{x=a/2} = 2.9$ kNm.

For the unloaded side the resulting bending moment due to the asymmetrical load is equal to:

$$M_{x=a/2} = -3.24 + 2.9 - 0.04 = -0.38 \text{ kNm}$$

The minimum normal force acting for x = -½ a at the unloaded side follows from:

$$N_{x=a/2} = -H\cos\beta - V_x\sin\beta + S\cos^2\beta + S\sin^2\beta = -7.2 \times \cos\beta - 1.8 \times \sin\beta + 6.6 = -0.8 \text{ kN}$$

For the loaded side the resulting bending moment due to the asymmetrical load is equal to:

$$M_{x=a/2} = +3.24 - 2.9 - 0.04 = +0.3 \text{ kNm},$$

For x = ½ a the maximum normal force is equal to:

$$N_{x=a/2} = -H\cos\beta - V_x\sin\beta - S\cos^2\beta - S\sin^2\beta = -7.2 \times \cos\beta - 1.8 \times \sin\beta - 6.6 = -14.0 \text{ kN}$$

Stresses

The stresses acting in the concrete, due to the normal forces and bending moments according to the Theory of Elasticity follow from: $\sigma_c = \dfrac{N E_c}{AE} \pm \dfrac{M z E_c}{EI}$

With: z= ½ ×110 = 55 mm; AE = 2.0×10^9 Nmm^{2}; $E_c = 2.7 \times 10^4$ MPa; EI= 2.8×10^{12} Nmm^{2T}

Table 6.2 shows for the strengthened vault the forces, bending moments and stresses due to the permanent and asymmetric load for x = ½ a.

Load	H [kN]	V [kN]	$N_{x=a/2}$ [kN]	$\sigma_c = N \times E_c / AE$ [MPa]	M [kNm]	$\sigma_c = M \times z \times E_c / EI$ [MPa]
perm. Load	29.9	15.25	-30.7	- 0.41	-0.19	± 0.10
sym.live load	14.4	7.20	-14.7	- 0.20	- 0.08	± 0.04
asym.live load unloaded side	7.2	1.80	- 0.8	- 0.01	- 0.38	± 0.20
asym.live load loaded side	7.2	5.40	-14.0	- 0.19	+0.30	± 0.16

TABLE 6.2 Forces, bending moments and stresses due to the permanent load and the live load.

Conclusions

Comparing table 6.1 and 6.2 shows that the strengthening with the diagonals reduces the bending moments and stresses. Especially for vaults, subjected to an asymmetrical live load, the strengthening with diagonals reduces the bending moments due to this load substantially. For the described vault the normal forces compensate for the better part the tensile bending stresses.

Calculation with computer program

For a two-hinged vault strengthened with two diagonals a calculation is made with a computer program, Matrixframe, with ties Ø25 and diagonals Ø100-4. The center of the coordinates is positioned at the crown. The nodes are numbered from the top to the supports. The following table shows the results for the permanent load and the asymmetrical live load acting at the right side.

The computer calculation shows that due to the strengthening the bending moments are quite small. The chosen length of the elements of the curved vault effects the bending moments. Due to the facetting the bending moments are increased with a bending moment. Increasing the number of the nodes and decreasing the length of the elements will decrease these bending moments.

The length of the elements is equal to dx = 0.9 m. For the permanent load this bending moment is at maximum equal to:

$$M = {}^{1}/_{8}\, q\, ds^2 = {}^{1}/_{8} \times 2.0 \times 0.9^2 = 0.2 \text{ kNm.}$$

For the permanent load the thrust is equal to H = 29.4 kN. At the top the normal force is equal to H = 29.7 kN. Due to the tensile forces S = + 0.3 kN acting at the diagonals the normal force acting at the top is smaller than the thrust.
For the three hinged vault the thrust is equal to H = 29.9 kN. For the design of a two hinged vault the thrust can be approached by assuming a third hinge at the top.
For the two hinged vault the bending moment acting at the top is equal to M = 0.51 kNm. According to the analyse for a not strengthened two hinged vault the bending moment at the top is for x/a = 0 equal to: $M_{x=0} = -{}^{1}/_{64}\, q.a^2$. Substituting q = 0.118 × 2.0 kN/m gives M = 0.19 kNm, this bending moment is smaller than the bending moment calculated with the computer for the two hinged vault strengthened with diagonals. Due to the strengthening the vault is subjected to an upward force equal to F = 0.38 kN. Due to this force the bending moment acting at the top is increased.
For the asymmetrical load the thrust is equal to H = 7.2 kN and the forces acting in the ties are S = + 6.7 kN and S = − 6.8 kN. The asymmetrical load can be considered as composed of a symmetrical load q = 0.5 kN/m and an antimetrical load q = 0.5 kN. Due to the symmetrical load q = 0.5 kN/m both diagonals are subjected to a force: S = -0.27 × 0.5/2.0 = -0.07 kN. Due to the antimetrical load the diagonals are subjected to a force S = 6.75 kN.
The force acting at the diagonals is, according to the analysis for the three hinged vault, equal to S = 6.6 kN. This force is slightly smaller than the force acting at the diagonals according to the computer calculation. Both calculations match well.

Node	X coord.	Y- coord.	Member	load case	moment	normal force	shear force
1	0	0	1-2	perm.	0.18	-29.7	0.94
2	0.9	0.028	2-3		0.18	-29.9	0.93
3	1.8	0.112	3-4		0.20	-30.2	0.93
4	2.7	0.253	4-5		0.25	-30.5	0.97
5	3.6	0.45	5-6		0.32	-31.0	0.96
6	4.5	0.703	6-7		0.36	-31.7	0.88
7	5.4	1.012	7-8		0.36	-32.4	0.92
8	6.3	1.378	8-9		0.29	-33.2	1.03
9	7.2	1.8	9-18, thrust			+29.4	
10	-0.9	0.028	1-9, diagonal			+ 0.3	
11	-1.8	0.112	1-2	asym. load	0.18	-13.7	0.60
12	-2.7	0.253	2-3		0.30	-13.8	0.57
13	-3.6	0.45	3-4		0.37	-13.9	0.50
14	-4.5	0.703	4-5		0.40	-14.4	0.46
15	-5.4	1.012	5-6		0.36	-14.7	0.49
16	-6.3	1.378	7-8		0.29	-15.0	0.51
17	7.2	1.8	8-9		0.18	-15.4	0.54
18	0	1.8	9-18, thrust			+ 7.2	
			1-9 diagonali			+ 6.7	
			1-17, diagonal			- 6.8	

TABLE 6.3 Output calcularion computerprogram for the two-hinged vault strengthened with diagonals

Approach

For the design of vaults, strengthened with diagonals, the force S acting at the diagonals can be approached for an asymmetrical load q by assuming a virtual hinge at x = ½ a. The force S follows for the anti-metrical load q', with q' is half of the asymmetrical load q, from:

$$S \times \tfrac{1}{4} f \cos\beta = q' a^2/8 \quad \rightarrow \quad S = \frac{{}^1/_2\, q' a^2}{f \cos\beta}$$

§ 6.7 Vaults strengthened with diagonal ties

Probably some of the remaining Fusee Céramique vaults made halfway the twentieth century have to be strengthened to meet the present demands. Architecturally it will be interesting to strengthen these structures with thin elements. However a slender tie, subjected to a compressive normal force, will fail by buckling. The previous chapters showed that due to asymmetrical loads the diagonals running from the top to the supports are subjected to a tensile force at the loaded part of the vault and subjected to a compressive force at the unloaded part of the vault. For vaults,strengthened with diagonal ties, the compressed tie can not resist any compressive load and has to be removed from the scheme of the structure. Thus the vault ,subjected to an asymmetrical load, is schemed as a structure strengthened with only one tie at the loaded side. The following analyses show the effect of the strengthening with slender diagonals not able to resist normal forces for an equally distributed and a linearly increasing load acting asymmetrically at the structure.

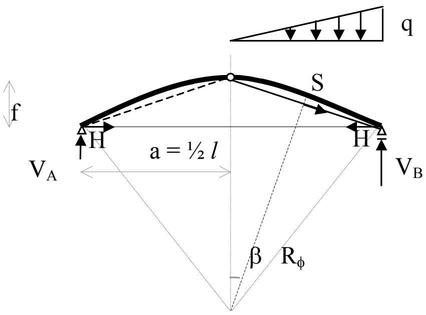

FIGURE 6.22 Parabolic vault strengthened with ties subjected to a linearly increasing asymmetrical load.The dotted tie is compressed and removed from the scheme

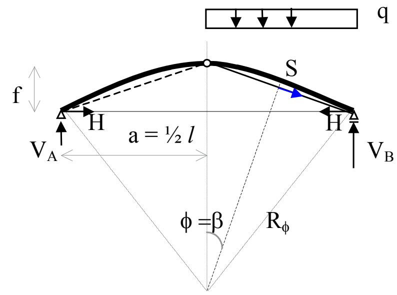

FIGURE 6.23 Parabolic vault strengthened with ties subjected to an asymmetrical load. The dotted tie is compressed and removed from the scheme

Parabolic vault, subjected to a linearly increasing load

The vault is subjected to an asymmetric equally distributed load q acting at the right half of the vault. The minimum and maximum vertical reaction are respectively equal to:

$V_a = {}^1/_{12}\, q\, a$ and $V_b = {}^5/_{12}\, q\, a.$

The thrust is calculated with the equilibrium of moments round the centre. Assume a virtual hinge is made at the top, then the bending moment acting on the top is zero. For the left side of the vault the equilibrium of bending moments around the top is equal to:

For the left side: $M = H f - V_a a = 0 \quad \rightarrow \quad H = {}^1/_{12} q a^2/f$ [6.19]

To calculate the force S the structure is made statically determinate by introducing an imaginary hinge at x = ½ a. As showed before the effect of the virtual hinge is small.

The bending moment follows from: $M_x = H y + V_a x - (½ q x^2/a) \times {}^1/_3 x$

For x = ½ a the bending moment is:

$$M_x = \frac{{}^1/_{12}\, q\, a^2 f \times ¼\, a^2}{f\, a^2} + {}^1/_{12}\, q\, a \times ½\, a - {}^1/_{48}\, q\, a^2 = {}^1/_{24}\, q\, a^2$$

The force S follows from:

$$S \times ¼ f \cos\beta = q\, a^2/24 \quad \rightarrow \quad S = \frac{q\, a^2}{6 f \cos\beta} \qquad [6.47]$$

For the left side the bending moment at a distance x from the centre follows from: $M_x = H y - V_a x$

Substituting H, V_a, gives: $M_x = \frac{q\, a^2 f x^2}{12 f a^2} - \frac{q\, a\, x}{12}$

The bending moment is at maximum for dM/dx = 0, differentiating M_x gives:

$$M_x = q\,x/6 \;\; - q\,a/12 = 0 \quad \rightarrow \quad x = ½\,a$$

For x = ½ a the bending moment is: $$M_x = \frac{q\,a^2}{48} - \frac{q\,a^2}{24} = -\frac{q\,a^2}{48} = 0.0208\,q.a^2$$

For the not-strengthened vault the bending moment follows from [6.20]: $M_{x\,max} = 0.044\,q\,a^2$
Thanks to the strengthening the bending moment, due to the asymmetrical linearly increasing load, is halved.

Parabolic vaultstrengthened with a tie, subjected to an asymmetric equally distributed load

The vault is subjected to an asymmetric equally distributed load q acting at a half of the vault. The structure is schemed as strengthened with the tensioned tie only; the compressed tie is taken away from the scheme.
The vertical reactions are respectively equal to: $V_a = ¼\,q\,a$ and $V_b = ¾\,q\,a$.

The thrust is calculated with the equilibrium of moments round the centre. Again the thrust follows from expression [6.8]. $$H = ¼\,q\,a^2/f$$

For two hinged low-rise vaults the vertical forces, bending moments and thrust do not vary much form the forces and bending moments acting on a three-hinged vault. To calculate the force S the structure is made statically determinate by introducing an imaginary hinge at x = ½ a. The bending moment follows from: $$M_x = H\,y + V_a\,x - ½\,q\,x^2$$
For x = ½ a, the bending moment is:

$$M_x = \frac{¼\;q\,a^2\,f \times ¼\;a^2}{f\,a^2} + ¼\,q\,a \times ½\,a \;-\; ½\,q \times ¼\,a^2 = q\,a^2/16$$

The force S follows from: $$S \times ¼\,f\cos\beta = q\,a^2/16 \quad \rightarrow \quad S = \frac{¼\,q\,a^2}{f\cos\beta} \qquad [6.48]$$

For the left side the bending moment at a distance x from the centre follows from: $M_x = H\,y\; - V_a\,x$

for x = ½ a the bending moment is: $$M_x = \frac{¼\;q\,a^2\,f \times ¼\,a^2}{f\;a^2} \;-\; ¼\,q\,a \times ½\,a = -\,q\,a^2/16$$

This bending moment is equal to the bending moment calculated for the not-strengthened vault.
Thus for this vault subjected to an asymmetrical equally distributed load the bending moment at the unloaded side is still equal to the bending moment acting in the not strengthened vault.

Tensioning the diagonals

At the loaded side the diagonal is tensioned but at the unloaded side the diagonal is compressed.
The bar can be constructed as a tie only in case the diagonal is subjected to a tensile force due to the dead load, which is larger than the compressive force due to the asymmetric live load. If the tension in the diagonal due to the dead load is insufficient to compensate for the compression, due to the asymmetrical load, then the diagonals must be tensioned to compensate the compressive force.
Due to a post-tensioning of the diagonals with a force P the vault will be subjected to bending moments: P e .For x = ½ a the distance between the parabolic vault and diagonal is equal to ¼ f , so the

eccentricity e follows from: $e = \frac{1}{4} f \cos\alpha$ with $\cos\alpha = a/(a^2+f^2)^{1/2}$ .Thus the bending moment acting at the vault is:

$$P \times \tfrac{1}{4} f \times a/(a^2+f^2)^{1/2} \qquad [6.49]$$

For a vault subjected to an asymmetrical load the force S follows from expression [6.48]:

$$S \times \tfrac{1}{4} f \cos\alpha = q\, a^2/16 \quad \rightarrow \quad S = \frac{\tfrac{1}{4} q\, a^2}{f \cos\beta} \qquad [6.48]$$

For the left side the bending moment at a distance x from the centre follows from: $M_x = H y - V_a x$
For x = ½ a the bending moment is:

$$M_x = \frac{\tfrac{1}{4} q\, a^2 f \times \tfrac{1}{4} a^2}{f\, a^2} - \tfrac{1}{4} q\, a \times \tfrac{1}{2} a = -q\, a^2/16$$

If the post tensioning force P is equal to the force S acting on the diagonal defined with expression (6.22) then the bending moment is:

$$M = \frac{a^2 q \times (1 + f^2/a^2)^{1/2}}{4 f} \times \frac{\tfrac{1}{4} f}{(1 + f^2/a^2)^{1/2}} \quad \rightarrow \quad M = \frac{q\, a^2}{16}$$

Due to the post tensioning the vault is subjected continuously to a bending moment equal to the bending moment caused by the asymmetrical live load.

Tensioning the diagonals with a strut

To reduce the bending moment due to the post-tensioning it is profitable to tension the diagonals by constructing a vertical strut halfway the span between the top of the vault and the tie with a length larger than f so the tie is inclined downward with an angle α. Due to the dead load the diagonals are stressed. The structure is statically indeterminate. As showed before for a vault subjected to concentrated force acting at the top the better part of the load will be resisted by the diagonals. For the design of the vault the assumption is made that the concentrated load is resisted by the diagonals only.

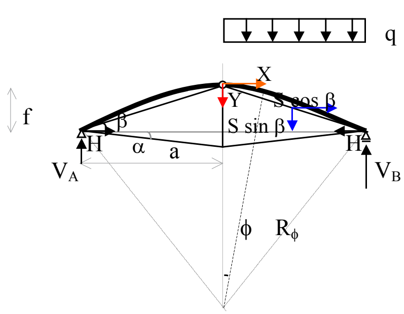

FIGURE 6.24 The parabolic vault is asymmetrical loaded. The tie is inclined downward. A vertical strut is made halfway the span between the tie and the vault to tension the diagonals..

Assume the force acting at the diagonal is S_d and the force acting on the tie is S_t. The force in the strut, F_{strut}, and the forces in the diagonals and tie follows from:

$$F_{strut} = S_d \sin\beta = S_t \sin\alpha \quad \rightarrow \quad S_d = S_t \sin\alpha/\sin\beta \qquad [6.50]$$

The thrust is resisted by the ties so: $H = S_d \cos\beta + S_t \cos\alpha$ [6.51]

Substituting [6.50] into [6.51]: $S_t\,[\sin\alpha/\tan\beta + \cos\alpha] = H$

The forces in the tie and diagonals are respectively:

$$S_t = \frac{H\ \tan\beta}{\cos\alpha\,[\tan\alpha + \tan\beta]} \qquad S_d = \frac{H \tan\alpha}{\cos\beta\,[\tan\alpha + \tan\beta]} \qquad [6.52]$$

Next the angle α is selected in such a way that the diagonals are stressed if the vault is subjected to an asymmetrical load.

Conclusions

Strengthening with slender diagonals reduces the bending moments due to a linearly increasing load but does not reduce the bending moments due to an asymmetrical load for the not loaded side. Nevertheless the ties will reduce the horizontal displacements due to the asymmetrical load or the wind load. Further the ties will reduce the buckling length too. So strengthening with ties will be helpful to increase the resistance of the vault. Tensioning the ties to prevent the ties being subjected to a compressive normal force is effective, if the vault is not loaded by the downward component of the post tensioning force. To prevent the vault being subjected to this component the tie between the supports has to be inclined and jointed with a vertical strut. The angle α has to be chosen carefully to tension the ties continuously, even in case the vault is subjected to an asymmetrical load.

§ 6.8 Parabolic trussed vault

For shallow vaults the distances between the vault and diagonals are quite small. For x = ½ a the distance between the vault and ties running from the supports to the top is only ¼ f. With a small strut this distance can be increased. The diagonals are connected with the tie running horizontally to increase the tensile forces. Integrating the diagonals and ties reduces the length of the elements. Structurally the trussed vault resembles the Polonceau truss.

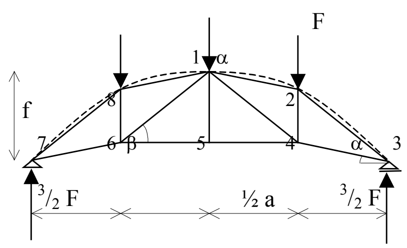

FIGURE 6.25 Trussed vault subjected to a symmetrically loading Caption

For the design and to show the distribution of the loads an analyse is made for several loads. The parameters of the trussed arch are the span l = 2 a, the rise f and the angle α and β for the web bars. The trussed vault is simplified into a statically determinate truss with straight hinged bars. The nodes are numbered clockwise.

The length of the strut at the centre is equal to ¾ f, the length of the other two struts is equal to ½ f.

The angle between the diagonals and the horizontal are equal to α and β with:

$$\tan\alpha = \tfrac{1}{4} f/(\tfrac{1}{2} a) = \tfrac{1}{2} f/a \quad \text{and} \quad \tan\beta = \tfrac{3}{4} f/(\tfrac{1}{2} a) = {}^{3}/_{2}\, f/a.$$

To simplify the analysis of the forces acting on the ties and struts the structure is assumed to be loaded only by concentrated forces acting on the joints. For every node the forces acting at the bars are defined with the vertical and horizontal equilibrium of the forces $\Sigma V = 0$ and $\Sigma H = 0$.

Equally distributed load

The truss is subjected to an equally distributed load, due to this load the nodes are subjected to concentrated forces equal to F, with $F = ½\, q\, a$. For every node the forces acting at the bars are defined with the vertical and horizontal equilibrium of the forces $\Sigma\varsigma = 0$. $\Sigma H = 0$. $\Sigma M = 0$.

Node 5: $\Sigma M_5 = 0$: $S_{45} = (^3/_2\, F\, a - F \times ½\, a)/(^3/_4\, f)$ → $S_{45} = {^4/_3}\, F\, a/f$

Node 2, $\Sigma M_2 = 0$: $S_{34} \cos\alpha = {^3/_2}\, F \times ½\, a/(½\, f) = {^3/_2}\, F\, a/f$ → $S_{34} = {^3/_2}\, F\, (a/f)/\cos\alpha$

$S_{34} \sin\alpha = {^3/_2}\, F\, (a/f) \times \tan\alpha$

Substituting $\tan\alpha = ½\, f/a$ gives: $S_{34} \sin\alpha = {^3/_2}\, F\, (a/f) \times ½\, f/a = {^3/_4}\, F$

Node 3, $\Sigma H_3 = 0$: $S_{23} \cos\beta = S_{34} \cos\alpha$

Substituting $S_{34} \cos a$ gives: $S_{23} = {^3/_2}\, F\, (a/f)/\cos\beta$ → $S_{23} \sin\beta = [{^3/_2}\, F\, (a/f)/\cos\beta] \times \sin\beta$

$S_{23} \sin\beta = {^3/_2}\, F\, (a/f) \times \tan\beta$

Substituting $\tan\alpha = {^3/_2}\, f/a$ gives: $S_{23} \sin\beta = {^3/_2}\, F\, (a/f) \times {^3/_2}\, f/a = {^9/_4}\, F$

Node 4, $\Sigma H_4 = 0$: $S_{45} = S_{34} \cos\alpha - S_{14} \cos\beta$ →

Substituting $S_{34} \cos\alpha$ and S_{45} gives: $S_{14} \cos\beta = [{^3/_2}\, Fa/f - {^4/_3}\, F\, a/f]$

$S_{14} \cos\beta = {^1/_6}\, F\, a/f$

$S_{14} \sin\beta = [{^1/_6}\, F\, (a/f)/\cos\beta] \times \sin\beta$ → $S_{14} \sin\beta = {^1/_6}\, F (a/f) \times \tan\beta$

Substituting $\tan\beta = {^3/_2}\, f/a$ gives: $S_{14} \sin\beta = {^1/_6}\, F\, (a/f) \times {^3/_2}\, f/a = ¼\, F$

Node 4: $\Sigma V_4 = 0$: $S_{24} = S_{14} \sin\beta + S_{34} \sin\alpha$

Substituting $S_{34} \sin\alpha = {^3/_4}\, F$ and $S_{14} \sin\beta = ¼\, F$ gives: $S_{24} = ¼\, F + ¼\, F = ½\, F$

Node 2, $\Sigma H_2 = 0$: $S_{12} \cos\alpha = S_{23} \cos\beta$

Substituting $S_{23} \cos\beta = {^3/_2}\, F \times a/f$: $S_{12} = ({^3/_2}\, F \times a/f)/\cos\alpha$

$S_{12} \sin\alpha = {^3/_2}\, F \times a/f \times \tan\alpha$

Substituting $\tan\alpha = {^1/_2}\, f/a$ gives: $S_{12} \sin\alpha = {^3/_2}\, F\, (a/f) \times {^1/_2}\, f/a = ¾\, F$

As all ties are tensioned only and thus not loaded in compression these elements can be dimensioned very slender.

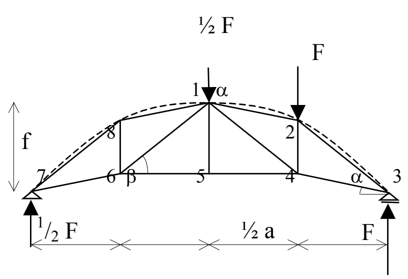

FIGURE 6.26 Trussed vault subjected to an asymmetrically loading

Trussed vault, asymmmetrically loaded.

The truss is subjected to asymmetric loading. Due to this load node 1 and 2 are subjected to respectively a load ½ F and F with $F = q\, a/2$. For every node the forces acting at the bars are defined with the vertical and horizontal equilibrium of the forces $\Sigma V = 0$, $\Sigma H = 0$, $\Sigma M = 0$, reactions $R_7 = ½\, F$ and $R_3 = F$.

Node 5: $\Sigma M_5 = 0$:	$S_{45} = (F \times a - F \times ½ a)/(^3/_4 f) \rightarrow$
	$S_{45} = {}^2/_3 F a/f$
Node 2: $\Sigma M_2 = 0$:	$S_{34} \cos\alpha = F ½ a/(½ f) \rightarrow$
	$S_{34} = F (a/f) / \cos\alpha$
	$S_{34} \sin\alpha = F (a/f) \times \tan\alpha$
Substituting $\tan\alpha = ½ f/a$ gives:	$S_{34} \sin\alpha = F (a/f) \times ½ f/a = ½ F$
Node 3: $\Sigma H_3 = 0$:	$S_{23} \cos\beta = S_{34} \cos\alpha$
Substituting $S_{34} \cos\alpha = F \times a/f$ gives:	$S_{23} = F (a/f) / \cos\beta$
	$S_{23} \sin\beta = F (a/f) \times \tan\beta$
Substituting $\tan\beta = {}^3/_2 f/a$ gives:	$S_{23} \sin\beta = F (a/f) \times {}^3/_2 f/a = {}^3/_2 F$
Node 4, $\Sigma H_4 = 0$:	$S_{45} = S_{34} \cos\alpha - S_{14} \cos\beta$
$S_{14} \cos\beta = F a/f - {}^2/_3 F a/f = {}^1/_3 F a/f \rightarrow$	$S_{14} \sin\beta = {}^1/_3 F (a/f) \times \tan\beta$
Substituting $\tan\beta = {}^3/_2 f/a$ gives:	$S_{14} \sin\beta = {}^1/_3 F (a/f) \times {}^3/_2 f/a = ½ F$
Node 4: $\Sigma V_4 = 0$:	$S_{24} = S_{14} \sin\beta + S_{34} \sin\alpha \rightarrow$
	$S_{24} = ½ F - ½ F = 0$
Node 2: $\Sigma H_2 = 0$:	$S_{12} \cos\alpha = S_{23} \cos\beta =$
Substituting $S_{23} \cos\beta = F \times a/f$ gives:	$S_{12} = F (a/f)/\cos\alpha$
	$S_{12} \sin\alpha = F (a/f) \times \tan\alpha$
Substituting $\tan\alpha = {}^1/_2 f/a$ gives:	$S_{12} \sin\alpha = F (a/f) \times {}^1/_2 f/a = ½ F$
Node 7, $\Sigma M_8 = 0$:	$S_{67} \cos\alpha = ½ F \times ½ a/(½ f) \rightarrow$
	$S_{67} = ½ F (a/f)/ \cos\alpha$
	$S_{67} \sin\alpha = {}^1/_2 F (a/f) \times \tan\alpha$
Substituting $\tan\alpha = ½ f/a$ gives:	$S_{67} \sin\alpha = {}^1/_2 F (a/f) \times ½ f/a = ¼ F$
Node 7: $\Sigma H_7 = 0$:	$S_{78} \cos\beta = S_{67} \cos\alpha \rightarrow$
Substituting $S_{67} \cos\alpha = ½ F \times a/f$:	$S_{78} = ½ F (a/f)/ \cos\beta$
	$S_{78} \sin\beta = ½ F (a/f) \times \tan\beta$
Substituting $\tan\beta = {}^3/_2 f/a$ gives:	$S_{78} \sin\beta = ½ F (a/f) \times {}^3/_2 f/a = {}^3/_4 F$
Node 6: $\Sigma H_6 = 0$:	$S_{56} = S_{67} \cos\alpha - S_{16} \cos\beta \rightarrow$
	$S_{16} \cos\beta = ½ F (a/f) - {}^2/_3 F (a/f) \rightarrow$
	$S_{16} \cos\beta = -{}^1/_6 F a/f$
	$S_{16} \sin\beta = {}^1/_6 F (a/f) \times \tan\beta$
Substituting $\tan\beta = {}^3/_2 f/a$ gives:	$S_{16} \sin\beta = {}^1/_6 F (a/f) \times {}^3/_2 f/a = ¼ F$
$\Sigma V_6 = 0$: $S_{68} = S_{16} \sin\beta + S_{67} \sin\alpha \rightarrow$	$S_{68} = ¼ F + ¼ F = ½ F$
Node 8, $\Sigma H_8 = 0$:	$S_{18} \cos\alpha = S_{78} \cos\beta = ½ F \times a/f \rightarrow$
	$S_{18} = [½ F a/f]/ \cos\alpha$
	$S_{18} \sin\alpha = ½F (a/f) \times \tan\alpha$
Substituting $\tan\alpha = {}^1/_2 f/a$ gives:	$S_{18} \sin\alpha = ½ F (a/f) \times {}^1/_2 f/a = ¼ F$

Due to the concentrated load the tie S_{16} is compressed, $S_{16} = -[{}^1/_6 F a/f]/\cos\beta$. Probably this tie is tensioned if the equally distributed load is larger than the concentrated load. Comparing the forces acting in the ties for the equally distributed load and the concentrated load shows that the tie is tensioned if the concentrated load due to the live load is smaller than the concentrated load due to the permanent load.

Concentrated load acting at the top

The truss is subjected to concentrated load F acting at the top. For every node the forces acting at the bars are defined with the vertical and horizontal equilibrium of the forces $\Sigma V = 0$; $\Sigma H = 0$; $\Sigma M = 0$.

Node 5: $\Sigma M_5 = 0$: $S_{45} = {}^1/_2\, F\, a/({}^3/_4\, f)$ $\rightarrow$ $S_{45} = {}^2/_3\, F\, a/f$

Node 2: $\Sigma M_2 = 0$: $S_{34} \cos\alpha = {}^1/_2\, F \times ½\, a/(½\, f)$ $\rightarrow$

$S_{34} = {}^1/_2\, F\, (a/f)/\cos\alpha$

$S_{34} \sin\alpha = {}^1/_2\, F\, (a/f) \times \tan\alpha$

Substituting $\tan\alpha = ½\, f/a$ gives: $S_{34} \sin\alpha = {}^1/_2\, F\, (a/f) \times ½\, f/a = {}^1/_4\, F$

Node 3: $\Sigma H_3 = 0$: $S_{23} \cos\beta = S_{34} \cos\alpha$

Substituting: $S_{34} \cos\alpha = {}^1/_2\, F \times a/f$ $S_{23} = {}^1/_2\, F\, (a/f)/\cos\beta$

$S_{23} \sin\beta = {}^1/_2\, F\, (a/f) \times \tan\beta$

Substituting: $\tan\beta = {}^3/_2\, f/a$ $S_{23} \sin\beta = {}^1/_2\, F\, (a/f) \times {}^3/_2\, f/a = {}^3/_4\, F$

Node 4: $\Sigma H_4 = 0$: $S_{45} = S_{34} \cos\alpha - S_{14} \cos\beta$ $\rightarrow$

$S_{14} \cos\beta = {}^1/_2\, F\, a/f - {}^2/_3\, F\, a/f$ $\rightarrow$ $S_{14} \cos\beta = -{}^1/_6\, F\, a/f$

$S_{14} \sin\beta = {}^1/_6\, F\, (a/f) \times \tan\beta$

Substituting: $\tan\beta = {}^3/_2\, f/a$ $S_{14} \sin\beta = {}^1/_6\, F\, (a/f) \times {}^3/_2\, f/a = ¼\, F$

Node 4: $\Sigma V_4 = 0$: $S_{24} = S_{14} \sin\beta + S_{34} \sin\alpha$ $\rightarrow$

$S_{24} = ¼\, F + ¼\, F = ½\, F$

Node 2: $\Sigma H_2 = 0$: $S_{12} \cos\alpha = S_{23} \cos\beta = {}^1/_2\, F\, a/f$ $\rightarrow$

$S_{12} = {}^1/_2\, F\, (a/f)/\cos\alpha$

$S_{12} \sin\alpha = {}^1/_2\, F\, (a/f) \times \tan\alpha$

Substituting $\tan\alpha = {}^1/_2\, f/a$ gives: $S_{12} \sin\alpha = {}^1/_2\, F\, (a/f) \times {}^1/_2\, f/a = ¼\, F$

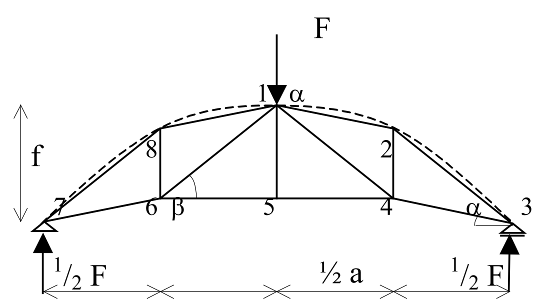

FIGURE 6.27 Trussed vault subjected to a concentrated force acting at the top

Due to the concentrated load the tie S_{14} is compressed: $S_{14} = -{}^1/_6\, F\, (a/f)/\cos\beta$ .Probably this tie is also tensioned if the equally distributed load is larger than the concentrated load. Comparing the forces acting in the ties for the equally distributed load and the concentrated load shows that the tie is tensioned if the concentrated load due to the live load is smaller than the concentrated load due to the permanent load.

Table 6.4 shows the results for the parameters, F, a, f, α and β. For low rise vaults cos α is approximately equal to 1, then a reduction of the ratio f/a will increase the forces acting on the elements nearly linearly. Trussed vaults are visually very transparent, architecturally these structures emphasizes the spatiality of the inner space. Due to the convex form the vertical strut running from the crown to the tie is smaller than the rise. Thus the normal forces acting on the vault and bars are increased due to due to the reduction of the lever arm. The effect of the reduction of the lever arm can be studied by varying the ratio f/a. If the live load is larger than the permanent load it can be profitable to increase the angle α.

Member	F_1	F_1, F_2, F_8	asym. ½ F_1, F_2
S_{12} =	$-\frac{F a}{2 f \cos\alpha}$	$-\frac{3 F a}{2 f \cos\alpha}$	$-\frac{F a}{f \cos\alpha}$
S_{23} =	$-\frac{F a}{2 f \cos\beta}$	$-\frac{3 F a}{2 f \cos\beta}$	$-\frac{F a}{f \cos\beta}$
S_{14} =	$-\frac{F a}{6 f \cos\beta}$	$\frac{F a}{6 f \cos\beta}$	$\frac{F a}{3 f \cos\beta}$
S_{34} =	$\frac{\frac{1}{2} F a}{f \cos\alpha}$	$\frac{3 F a}{2 f \cos\alpha}$	$\frac{F a}{F \cos\alpha}$
S_{456} =	$^2/_3\, F a/f$	$^4/_3\, F a/f$	$^2/_3\, F a/f$
S_{24} =	½ F	½ F	0
S_{18} =	$-\frac{F a}{2 f \cos\alpha}$	$-\frac{3 F a}{2 f \cos\alpha}$	$-\frac{F a}{2 f \cos\alpha}$
S_{78} =	$-\frac{\frac{1}{2} F a}{f \cos\beta}$	$-\frac{3 F a}{2 f \cos\beta}$	$-\frac{F a}{2 f \cos\beta}$
S_{16} =	$-\frac{F.a}{6 f \cos\beta}$	$\frac{F a}{6 f \cos\beta}$	$-\frac{F.a}{6 f \cos\beta}$
S_{67} =	$\frac{\frac{1}{2} F a}{f \cos\alpha}$	$\frac{3 F a}{2 f \cos\alpha}$	$-\frac{F a}{2 f \cos\alpha}$
S_{68} =	½ F	½ F	½ F

TABLE 6.4 Forces acting in the trussed vault with $\tan\alpha = \frac{1}{2} f/a$ and $\tan\beta = {}^3/_2\, f/a$

§ 6.9 Conclusions

This chapter shows the effect of strengthening parabolic vaults with ties. Several concepts are analysed. To strenghten an existing vault adding a truss composed of two diagonals is effective. The diagonals will not increase the dead weight significantly. For vaults subjected to an asymmetric live load the diagonals can be subjected to compression, consequently the stiffness of these diagonals must be increased to prevent buckling. Post-tensioning the diagonals is not very effective. Due to the post-tensioning the vault is subjected to a concentrated force acting at the centre causing bending moments. By preference the diagonals are tensioned with a vertical strut constructed between the vault and the tie. The tie between the supports is inclined downward to create an upward force at the top, this force will tension the diagonals continuously. For vaults subjected to a substantial live load trussed vaults are very effective. Generally the compressive forces acting at the ties due to asymmetrical loads will be compensated by the tensile forces due to the permanent load, so the ties of a trussed vault can be very slender. For trussed vaults the struts and web bars have to be connected to the vault, this will raise the cost of construction, especially for the renovation of an existing vault. Consequently for a renovation it is advisable to strengthen these vaults with simple trusses composed of two diagonals.

7 Strengthening the Fusée Céramique roof of building Q in Woerden

It was demonstrated in chapter 5 that the fusée roof of building Q had to be strengthened to resist the loads. Chapter 6 shows how the structure can be strengthened. Two alternatives are developed: the structure is strengthened with two diagonals running from the supports to the top and the structure is strengthened as a trussed arch. The vault has a span of l = 2 a = 19.8 m centre to centre. The rise f of the roof is equal to 2.48 m. The ratio of the rise and span is equal to: $f/l = {}^{1}/_{8}$. According to the code of 1955 for roofs the live load was equal to p_e = 0.5 kN/m². The permanent load is equal to: p_g = 2.4 kN/m². For a segment of the roof with width of 1.0 m the areas and the second moment of the area of the concrete, fusées and steel are shown in table 7.1.

		Area [mm²]		Second moment of the area [mm⁴]	YOUNG'S MODULUS [MPA]
Fusées:	A_f =	24.19×10^3	I_c =	160.97×10^6	$E_f = 1.7 \times 10^4$
Concrete:	A_c =	74.708×10^3	I_f =	15.12×10^6	$E_c = 2.1 \times 10^4$
Rebars:	A_s =	2×279	I_s =	1.18×10^6	$E_s = 2.1 \times 10^5$

TABLE 7.1 Area and second moment of the area of the fusées, concrete and steel

The stiffness AE and EI are calculated by multiplying the area and second moment of the area with the Young's modulus. To simplify the calculation a ratio n_f and n_s is introduced with: $n_f = E_f / E_s = 0.81$ and $n_s = E_s / E_c = 10$, EA and EI are calculated with respectively [6.25] and [6.26]:

$$EA = E_c (A_c + n_f A_f + n_s A_s) = 2.1 \times 10^9 \text{ N}$$

$$EI = E_c (I_c + n_f I_f + n_s I_s) = 3.89 \times 10^{12} \text{ Nmm}^2$$

To show the effect of the strengthening methods for building Q three alternatives are analysed, the non-strengthened vault, the vault strengthened with diagonals and the trussed vault.

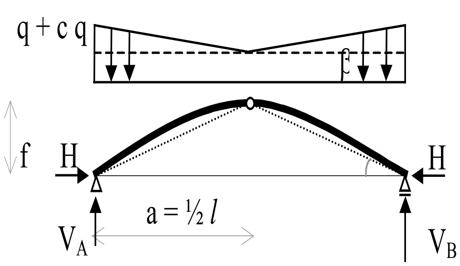

FIGURE 7.1 Vault, strenghtened with two diagonals, subjected to the permanent load

§ 7.1 Non strengthened vault

The Forces acting at the vault were calculated with the Theory of Elasticity. The vault was schematized as an arch resting on two simple supports. The stiffness of the supports was neglected, actually the ties will lengthen so the supports will move sideward. The thrust was calculated with the equilibrium of bending moment around the top. Actually the bending moment at the top is not zero, the effect of this assumption is negligible as will be shown later.

Permanent load

The vault is subjected to the dead load at the top this load is equal to q_g = 2.4 kN/m². The calculation is made for a width of 1.0 m. The span is equal to l = 19.8 m, half of the span is equal to a = ½ l = 9.9 m. Due to the curvature of the vault the permanent load increases from the top to the supports, with:

$$q_{max} = q\ ds/dx = q\,[1+ (y')^2]^{½} \quad \text{with: } y' = 2\,f\,x/a^2$$

Substituting y' gives the for the increasing surface load: $q_{max} = q\ [1+(2\,f\,x/a^2)^2]^{½}$

For x = a the maximum load is equal to: $q_{max} = q\ (1+ 4\,f^2/a^2)^{½}$

With f = 2.48 and a = 9.9 m the maximum load is equal to:

$$q_{max} = q\,(1+ 4 \times 2.48^2/9.9^2)^{½} = 1.1185\,q$$

The increase in the load isn't large, to analyse this structure the increasing load is modelled as a combination of an equally distributed load q and a linearly increasing load q_{inc}:

$$q_{max} = q + q_{inc} = (1 + 0.1185) \times 2.4 \text{ kN/m.}$$

For the eqully distributed load q = 2.4 kN/m the forces are calculated as follows:
The vertical and horizontal reaction force acting on the supports are respectively:

$$V_A = V_B = q_g\,a = 2.4 \times 9.9 = 23.76 \text{ kN}$$

$$H = \frac{q_g\,a^2}{2\,f} = \frac{2.4 \times 9.92}{2 \times 2.48} = 47.424 \text{ kN}$$

For the increasing load q_{inc} = 0.1185 × 2.4 kN/m the forces and bending moments are calculated as follows:
The vertical reaction acting at a support is equal to:

$$V_A = V_B = ½\,q_{inc} \times a = 0.1185 \times 2.4 \times 9.9/2 = 1.41 \text{ kN}$$

The thrust H acting at the supports follows from the equilibrium of the moments around the centre at the top.

$$H \times f - V\,a + {}^1/_3\,q_{inc}\,a^2 = 0 \quad \rightarrow$$

$$H = {}^1/_6\,q_{inc}\,a^2/f = {}^1/_6\ 0.1185 \times 2.4 \times 9.9^2/\,2.48 = 1.87 \text{ kN}$$

The bending moment M_x at a distance x from the top follows from [6.12]:

$$M_x = {}^1/_6\,q_{inc}\,(x^2 - x^3/a)$$

The bending moment is maximum when $x = {}^2/_3\ a$, substituting $x = {}^2/_3\ a$ into the expression for M_x results in:

$$M_x = {}^1/_6\, q_{inc}\ [\, ({}^2/_3\, a)^2 - ({}^2/_3\, a)^3/a\,] \quad \rightarrow \quad M_x = {}^2/_{81}\, q_{inc}\ a^2$$

$$M_{max} = {}^2/_{81} \times 0.1185 \times 2.4 \times 9.9^2 = 0.69 \text{ kNm}$$

Due to increasing load the vault is subjected to: $V = 23.76 + 1.41 = 25.2$ kN
$H = 47.424 + 1.87 = 49.1$ kN

The normal force follows from: $N = (H^2 + V^2)^{½}$, for $x = ½\, a = 4.95$ m the normal force is:

$$N = [H^2 + (\, ½\, q\, a + {}^1/_8\, q_{inc}\, a)^2]^{0.5}$$

$$N = [49.1^2 + (2.4 \times 9.9/2 + 0.1185 \times 2.4 \times 9.9/8)^2]^{0.5} = 50.6 \text{ kN}$$

Variable load

The vault is subjected to a live load $q_e = 0.5$ kN/m^2. Due to the symmetrical live load the vertical and horizontal reaction force acting on the supports are respectively:

$$V_A = V_B = q_e\, a = 0.5 \times 9.9 = 4.95 \text{ kN}$$

$$H = \frac{q_e\, a^2}{2f} = \frac{0.5 \times 9.9^2}{2 \times 2.48} = 9.9 \text{ kN}$$

For the permanent surface load and variable load the vertical and horizontal reaction force acting on the supports are respectively:

$$V = 25.2 + 4.95 = 30.2 \text{ kN},\ \ H = 49.1 + 9.9 = 59.0 \text{ kN};$$

For $x = ½\, a = 4.95$ m the normal force follows from: $N = [H^2 + (\, ½\, q\, a + {}^1/_8\, q_{inc}\, a + ½\, q_e\, a)^2\,]^{0.5}$

$$N = [\, 59.0^2 + (2.4 \times 9.9/2 + 0.1185 \times 2.4 \times 9.9/8 + 0.5 \times 9.9/2)^2\,]^{0.5} = 60.8 \text{ kN}$$

Asymmetric load

Due to an asymmetric load the vault is subjected to bending. Assuming that the vault is subjected to a live load acting on one side equal to $q_e = 0.5$ kN/m^2. The vertical and horizontal reaction forces acting on the supports are respectively:

$$V_A = ¼\, q_e\, a = ¼ \times 0.5 \times 9.9 = 1.24 \text{ kN}$$
$$V_B = ¾\, q_e\, a = ¾ \times 0.5 \times 9.9 = 3.71 \text{ kN}$$

$$H = \frac{q_e\, a^2}{4f} = \frac{0.5 \times 9.92}{4 \times 2.48} = 4.94 \text{ kN}$$

The bending moment is equal to: $M_o = \frac{q_e\, a^2}{16} = \frac{0.5 \times 9.92}{16} = 3.1$ kNm

For the permanent surface load and asymmetrical variable load the vertical and horizontal reaction force acting on the supports are respectively:

$V_B = 25.2 + 3.71 = 28.91$ kN

$H = 49.1 + 4.94 = 54.04$ kN

For $x = ½\,a = 4.95$ the shear force is equal to:

$$V = ½\,q_g\,a + {}^1/_8\,q_{inc} + ¼\,q_e\,a = 2.4 \times 9.9/2 + 0.1185 \times 2.4 \times 9.9/8 + 0.5 \times 9.9/4 = 13.5 \text{ kN}$$

For $x = ½\,a = 4.95$ the normal force due to the permanent surface load and the variable load is equal to:

$$N = [\,H^2 + (½\,q_g\,a + {}^1/_8\,q_{inc} + ¼\,q_e\,a\,)^2\,]^{0.5} = [54.04^2 + 13.5^2]^{0.5} = 55.7 \text{ kN}$$

§ 7.2 Vault strengthened with two diagonals

The vault is strengthened with two diagonals running from the top to the supports. The diagonals will decrease the bending moments for the asymmetrical live load and the surface load increasing from the top to the supports.

For the linearly increasing surface load the maximum bending moment is:

$$M_{max} = -0.01304 \times q_{inc}\,a^2 = 0.01304 \times 0.1185 \times 2.4 \times 9.9^2 = 0.36 \text{ kNm}$$

For the design of this structure the force S acting on the diagonal is approached with the expression described in chapter 6:

$$S_d = \frac{M_{x=a/2}}{¼\,f\cos\beta}$$

Substituting $\cos\beta = 1/(1+f^2/a^2)^{1/2}$:

$$S_d = \frac{M_{x=a/2}\,(1 + f^2/a^2)^{1/2}}{¼\,f}$$

For the linearly increasing load the bending moment is for $x = ½\,a$ equal to: $M_{x=a/2} = q\,a^2/48$

$$S_d = \frac{q\,a^2\,(1 + f^2/a^2)^{1/2}}{48 \times ¼\,f} = \frac{0.1185 \times 9.9^2 \times (1 + 2.48^2/9.9^2)^{1/2}}{48 \times ¼ \times 2.48} = 0.4 \text{ kN}$$

The strengthened vault subjected to the asymmetrical live load is not loaded by bending moments. The force acting at the diagonals S_d follows from:

$$S_d = \frac{M_{x=a/2}}{¼\,f\cos\beta} = \frac{M_{x=a/2}\,(1 + f^2/a^2)^{1/2}}{¼\,f}$$

For the anti-metrical load q' the bending moment is equal to: $M_{x=a/2} = q'\,a^2/8$ with $q' = ½\,q$, thus:

$$S_d = \pm\frac{{}^1/_2\,q\,a^2\,(1 + f^2/a^2)^{1/2}}{8 \times {}^1/_4\,f} = \pm\frac{{}^1/_2 \times 0.5 \times 9.9^2 \times (1 + 2.48^2/9.9^2)^{1/2}}{8 \times ¼ \times 2.48} = \pm 5.1 \text{ kN}$$

At the left side, which is not loaded by the asymmetrical live load, the diagonal is compressed. The diagonal must be so stiff that this element does not fail by buckling: $N_{buc} > n_{cr} N_d$. For the design of the diagonal it is advisable to apply a factor n ≥ 4 for the design load.

$$N_{buc} = \frac{\pi^2 EI}{(a/\cos\beta)^2} \geq n_{cr} N_d$$

Due to the permanent load the diagonal is tensioned, the load factor is equal to: $\gamma_g = 0.9$. Due to the asymmetrical load the diagonal is compressed, the load factor is equal to: $\gamma_e = 1.5$. The diagonal is subjected to a normal compressive force:

$$N_d = -(-0.9 \times 0.4 + 1.5 \times 5.1) = -7.29 \text{ kN.}$$

Substituting a = 9.9 m, cos β = 0.97, $E_s = 2.0 \times 10^5$ MPa, $n_{cr} = 4$, $N_d = 7.29$ kN gives:

$$N_{buc} = \frac{\pi^2 \times 2.0 \times 10^5 \times I}{(9900/0.97)^2} \geq 4 \times 7290 \text{ N} \qquad I \geq 1.54 \times 10^6 \text{ mm}^4,$$

The diagonal can be dimensioned quite slender , for example with a steel tube Ø101.6-4.5 with $I = 1.62 \times 10^6$ mm^4.

Post-tensioning

The diameter of the diagonal can be decreased if the diagonal is subjected to tensile forces only. Then the tensile force from the load has to exceed the compressive force caused by the asymmetric live load. If the tension in the diagonal due to the dead load is insufficient to compensate for the compression caused by the asymmetrical load then the diagonals can be post-tensioned. Due to a post-tensioning force P the vault is subjected to a bending moment M_p:

$$M_p = P\,e = P \times (\tfrac{1}{4} f \cos\alpha) = 5.1 \times (\tfrac{1}{4} \times 2.48 \times 0.97) = 3.1 \text{ kNm}$$

This bending moment is equal to the bending moment due to the asymmetrical live load, M = 3.1 kNm. So post-tensioning of the diagonals seems not very effective. Otherwise due to the post-tensioning of the diagonals both diagonals are tensioned and reduce the buckling length of the vault considerately.

For the ultimate state the loads must be multiplied with a load factor. For the post-tensioning the load factor is 1.0 so the bending moment is $M_{Pd} = 1.0 \times 3.1$ kNm. Further the post-tensioning does not generate second order moments. For a variable load the load factor is 1.5 , so the bending moment due to the asymmetrical live load is exclusive second order equal to $M_{ed} = -1.5 \times 3.1 = 4.65$ kNm. Consequently the post-tensioning reduces the second order effect and bending moments.

Tensioning the diagonals by lengthening of the strut at the centre

The diagonals can be tensioned by introducing an upward force acting at the top of the vault. This force can be created if tie is connected with a single strut at the centre with the top of the vault, see figure 7.2. Lengthening of the strut will move the tie downward. Thus the strut is compressed and the top of the vault is subjected to an upward vertical force. This force provides a tensile force acting on the diagonals. The angle between the horizontal line through the supports and the tie is called α.

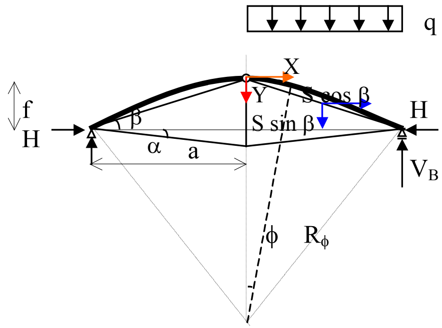

FIGURE 7.2 Parabolic vault, the diagonals are tensioned by the upward force in the strut

Assume the force acting at the diagonal is S_d and the force acting on the tie is S_t. The forces S_d and S_t are defined in chapter 6, expression [6.52].

$$S_t = \frac{H \tan\beta}{\cos\beta\,[\tan\alpha + \tan\beta]} \qquad S_d = \frac{H \tan\alpha}{\cos\alpha\,[\tan\alpha + \tan\beta]} \qquad [6.52]$$

Next the angle α is selected in such a way that the diagonal ties are stressed if the vault is subjected to the asymmetrical load. For S_d = 5.1 kN, H = 54.04 kN, $\tan\beta$ = 0.25 and $\cos\beta$ = 0.97 the tangent of α is defined with [6.52]:

$$\tan\alpha = \cos\beta\,[\tan\alpha + \tan\beta]\; S_d/H \;\rightarrow\; \tan\alpha \geq \frac{\cos\beta \times \tan\beta \times S_d / H}{(1-\cos\beta\; S_d/H)}$$

$$\tan\alpha \geq \frac{0.97 \times 0.25 \times 5.1/54.04}{(1-0.97 \times 5.1/54.04)} = 0.025$$

The sagging of the tie has to be 0.25 m at minimum. It is advisable to increase the sacking slightly. For a sagging of 0.25 m, $\tan\beta$ = 0.025, $\cos\beta$ = 1.0, $\sin\alpha$ = 0.243, H = 54.04 kN, the force acting in the tie and diagonal are respectively:

$$S_t = \frac{H \tan\beta}{\cos\beta\,[\tan\alpha + \tan\beta]} = \frac{54.04 \times 0.25}{1 \times [0.023 + 0.25]} = 49.1 \text{ kN}$$

$$S_d = \frac{H \tan\alpha}{\cos\alpha\,[\tan\alpha + \tan\beta]} = \frac{54.04 \times 0.023}{0.97 \times [0.023 + 0.25]} = 5.1 \text{ kN}$$

In this analysis the stiffness of the vault is neglected so actually the forces acting on the diagonals will be slightly smaller. The ties must be attached well to the vault. Due to the tensile forces the joints will be subjected to shear forces and tensile forces. It can be diffcult to attach bolts firmly with the Fusée Céramique vault. The thickness of the vault is small and the space between the fusées is only 10 mm. A bolt positioned in a fusée element will not be able to resist a substantial pulling force.

Buckling

As described in chapter 3 Moon et al [Moo07] researched the critical buckling load for parabolic pin-ended arches. For in-plane asymmetric buckling mode the critical buckling load for pin–ended arches is:

$$N_{cr\,asym} = \pi^2 EI/s^2 \qquad [3.13']$$

TOC

Thanks to the diagonals the vault is supported at the top, the buckling length of the vault is halved and equal to half the length of the vault between the top and supports: l_c = ½ s.The inplane asymmetric buckling load will be decisive in case the rise f meets the following condition:

$$f/a > 4.565 \times i/a \qquad [3.14]$$

The radius of gyration of the section is equal to: $i = (I/A)^{1/2}$ For: EA = 2.1 × 10^9 Nmm² and EI = 3.89 × 10^{12} Nmm² the radius of gyration of the section is equal to:

$$i = (I/A)^{1/2} = [3.89 \times 10^{12} / 2.1 \times 10^9]^{1/2} = 43 \text{ mm}$$

For the vault the rise f is 2.48 m. Snap through will be not decisive if the distance f' of the diagonal and the vault exceed:

$$f' > 4.565 \times 43 = 198 \text{ mm.}$$

The distance of the vault and the diagonal is equal to f' = ¼ f = 2.48/4 = 0.62 m. Consequently for this structure the asymmetric buckling load will be decisive. The length s of the vault between the top and support is equal to:

$$s = f(1 + \tfrac{1}{4} a^2/f^2)^{1/2} + \tfrac{1}{4} a^2/f \times \ln\{2 \times f/a + (4 f^2/a^2 + 1)^{1/2}\} \qquad [6.4]$$

Substituting f = 2.48 m, a = 9.9 m and a/f = 4:

$$s = 2.48 \times (1 + \tfrac{1}{4} \times 4^2)^{1/2} + \tfrac{1}{4} \times 9.9 \times 4 \times \ln\{\tfrac{1}{2} + (4/16 + 1)^{1/2}\} = 5.545 + 4.763 = 10.31 \text{ m}$$

With a buckling length equal to ½ s = 5.15 m the critical buckling load is equal to:

$$N_{cr\,asym} = \frac{\pi^2 EI}{(\tfrac{1}{2} s)^2} = \frac{\pi^2 \times 3.89 \times 10^{12}}{5150^2} = 1448 \times 10^3 \text{ N}$$

Asymmetrical load

For x = ½ × a the normal force is equal to N = 57.5 kN. Thus the ratio n of the buckling force and the normal force is: $n_{cr} = N_{cr}/N = 1448/57.5 = 25$

Generally a ratio of $n_{cr} \geq 5$ was recommended (for representative loads). For the full load as well as the asymmetrical load the ratio n_{cr} is larger than 5. Due to the diagonals the second order effect is reduced considerately, but time dependent effects and cracks will decrease the stiffness and the critical buckling force. The effect of this will be considered later.

Calculation of the normal and bending stress.

According to the Theory of Elasticity the normal stress due to the normal force acting on the section of the vault follows for the concrete, fusées and reinforcement from:

$$\sigma_x = \frac{N E_x}{EA_c m_{EA}} \qquad \text{with } m_{EA} = 1 + m_f E_c A_f + m_s E_c A_s \qquad [5.6]$$

Due to the dead load and the asymmetrical live load the structure is subjected to normal stresses and bending stresses. At a quarter of the span the normal force is equal to N = 57.5 kN.

$$m_{EA} = 1 + m_f E_c A_f + m_s E_c A_s = 1.34$$

$$E_c A_c m_{EA} = 2.1 \times 10^4 \times 74.708 \times 10^3 \times 1.34 = 2.1 \times 10^9 \text{ Nmm}^2$$

The stress in the concrete, due to the normal load, is: σ_c = 0.58 MPa

Thanks to the diagonals the vault is no longer subjected to a bending moment, but if the diagonals are post-tensioned then the vault is subjected to a bending moment, M_p = 3.2 kNm.

$$\sigma_c = \frac{M\,z\,E}{EI} = \frac{3.2 \times 10^6 \times 110/2 \times 2.1 \times 10^4}{3.89 \times 10^{12}} = 0.95 \text{ MPa}$$

The resulting stresses due to the bending and normal force are:

compressive stress: σ_c = -0.58 - 0.95 = - 1.53 MPa

tensile stress: σ_c = -0.58 + 0.95 = +0.37 MPa

For the concrete sections creep and shrinkage increases the tensile stresses, possible the vault will be cracked. The cracks will reduce the stiffness of the vault, consequently the critical buckling force is reduced.

Verification with computer program

For the structure strengthened with diagonals the deformations, forces and bending moments are calculated with a computer program (Matrixframe). The structure is subjected to the surface load Q = 2.4 kN/m increasing to the supports, A live load of q = 0.5 kN/m and the wind load. Table 7.2 shows the coordinates and permanent load.

Node	X	Z	Member	dy/dx	ds	Permanent load
1	-9.9	2.48				
2	-9.0	2.05	1-2	0.43	1.088	2.61
3	-8.0	1.62	2-3	0.38	1.07	2.57
4	-7.0	1.24	3-4	0.33	1.053	2.53
5	-6.0	0.91	4-5	0.28	1.038	2.49
6	-5.0	0.63	5-6	0.22	1.024	2.46
7	-4.0	0.41	6-7	0.18	1.016	2.44
8	-3.0	0.23	7-8	0.13	1.008	2.42
9	-2.0	0.1	8-9	0.07	1.002	2.41
10	-1.0	0.03	9-10	0.03	1.0	2.4
11	0	0	10-11	0	1.0	2.4
12	1.0	0.03	11-12	0	1.0	2.4
13	2.0	0.1	12-13	0.03	1.0	2.4
14	3.0	0.23	13-14	0.07	1.002	2.41
15	4.0	0.41	14-15	0.13	1.008	2.42
16	5.0	0.63	15-16	0.18	1.016	2.44
17	6.0	0.91	16-17	0.22	1.024	2.46
18	7.0	1.24	17-18	0.28	1.038	2.49
19	8.0	1.62	18-19	0.33	1.053	2.53
20	9.0	2.05	19-20	0.38	1.07	2.57
21	9.9	2.48	20-21	0.43	1.088	2.61

TABLE 7.2 : Coordinates of the vault

The wind load acting on the roof is calculated according to the Euro code NEN EN 1991-1-4-2005.

The city of Woerden is situated at the frontier between Zuid-Holland and Utrecht. The height of the structure is 7.55 m. For an urban area II, without adjacent buildings, the wind load is interpolated for respectively z = 7.0 m and z = 8.0 m, with for z = 7.0 m: $q_{z=7.0}$ = 0.75 kN/m² and for z = 8.0 m: $q_{z=8.0}$ = 0.79 kN/m² . Interpolation gives for z = 7.55: $q_{z=7.55}$ = 0.75 + 0.04 × 0.55 = 0.77 kN/m²

The coefficients for internal pressure are for overpressure c = + 0.2 and for under pressure c = -0.3.

The coefficients for the external wind load are:

area A:	windward side, sucking and pressure:	c= -1.2 and c = + 0.1
area B:	at the top, sucking;	c = -0.82
area C:	leeward side, sucking:	c = -0.4

Combining the internal and external wind pressure, two extreme wind loads arise, respectively with over and under pressure:

	over pressure	under pressure
A:	p = (-1.2 - 0.2) × 0.77 = -1.08	p = (+0.1 + 0.3) × 0.77 = + 0.31
B:	p = (-0.82- 0.2) × 0.77 = -0.79	p = (-0.82+ 0.3) × 0.77 = - 0.40
C:	p = (-0.4 - 0.2) × 0.77 = -0.46	p = (-0.4 + 0.3) × 0.77 = - 0.08

Snow loads

According to the NEN-EN 1991-1-3 the roof is subjected to a snow-load: $p_{sn} = u_3 \times 0.7$ kN/m².
The coefficient u_3 depends on the ratio of the rise versus the span.

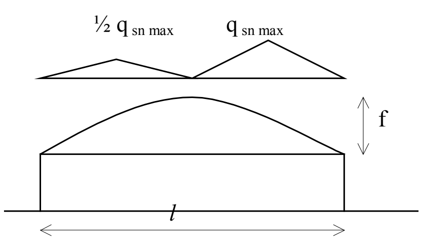

FIGURE 7.3 linearly increasing snowload acting on the vault with a maximum at x = a/2

Two alternatives are distinguished, the snow load can be equally distributed or linearly increasing,.
For an equally distributed load and a cylindrical roof with a ratio f/l = 0.125 the coefficient u_3 is equal to u_3 = 0.8: and p_{sn} = 0.8 × 0.7 = 0.56 kN/m².

For a linearly increasing snow load, with a maximum at ¼ l , u_3 follows from: $u_3 = 0.2 + 10 \times f/l$

For f/l = 0.125 the coefficient u_3 is equal to: u_3 = 0.2 + 10 × 0.125 = 1.45.
The maximum load acting at the roof is: $p_{sn\ max}$ = 1.45 × 0.7 = 1.02 kN/m².

Output

The following table shows the results of the analysis with the computer program Matrixframe.

Due to the diagonals the bending moments are much smaller than for the non-strengthened vault. The maximum bending moment resulting from the wind load is equal to M = 1.09 kNm. This moment is about half the bending moment acting in the non-strengthened vault, M = 3.1 kNm, due to the live load.

Member	Load:	Nx	Vz	M
4	permanent load	-51.22	1.31	0.50
	Live load left	-10.31	0.24	0.13
	snow load	- 5.81	0.29	0.10
	wind over pressure	20.51	0.85	0.80
	wind under pressure	0.57	0.51	1.09
6	permanent load	-49.78	1.44	0.37
	Live load left	-10.01	0.31	0.15
	snow load	- 6.57	0.23	0.20
	wind over pressure	20.55	0.68	0.30
	wind under pressure	0.49	0.89	0.70
10	permanent load	-48.36	1.42	0.29
	Live load left	- 9.72	0.34	0.06
	snow load	- 6.38	0.22	0.15
	wind over pressure	20.55	0.72	0.34
	wind under pressure	0.53	1.44	0.25
22 tie	permanent load	+48.19		
	Live load left	+4.96		
	snow load	+7.39		
	wind over pressure	-15.12		
	wind under pressure	- 6.24		
24, diagonal	permanent load	-0.12		
	Live load left	+4.90		
	snow load	- 1.05		
	wind over pressure	- 4.19		
	wind under pressure	+ 6.28		
25, diagonal	permanent load	- 0.12		
	Live load left	- 5.05		
	snow load	+ 5.31		
	wind over pressure	+ 2.83		
	wind under pressure	+ 2.44		

TABLE 7.3 Results of the calculations with the computerprogram: normal forces, shear forces and bending moments

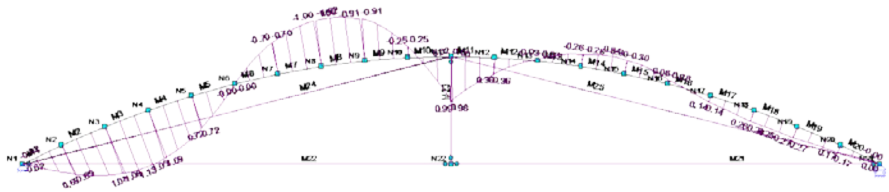

FIGURE 7.4 Bending moments due to the windload underpressure

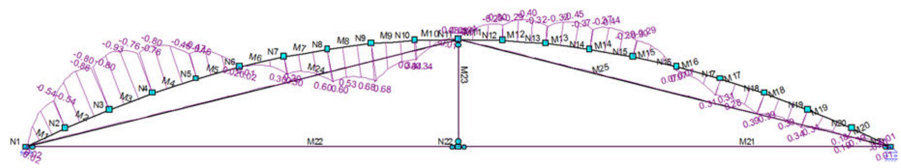

FIGURE 7.5 Bending moments due to the windload overpressure

Post-tensioning

Due to the asymmetrical load and the wind load the diagonals are tensioned or compressed. To resist a compressive force the diagonals must be stiff enough to resist buckling or must be post-tensioned. The maximum compressive force acting at the diagonal is −5.05 kN. The post-tensioning of the diagonal has to be at least equal to this compressive force, so P = 5.05 kN. Due to a post-tensioning force P the vault is subjected to a bending moment M_P:

$$M_P = P \times e = P \times (^1/_4\, f) \times \cos\alpha = 5.05 \times (2.48/4) \times 0{,}97 = 3.04\ \text{kNm}$$

Tensioning the diagonals with the strut

To prevent a compressive load acting on the diagonals tie is constructed with a declination of 0.25 m. Due to the trust the strut halfway the span is subjected to a normal forces acting upward. This force is solved into to components parallel to the diagonals, loading the diagonals with a tensile force.

Due to the permanent load the diagonals are subjected to a tensile normal force N= 2.62 kN. But due to the asymmetrical live load or the wind load one of the diagonals is subjected to a compressive normal force of respectively − 4.79 kN and − 4.98 kN. Thus the normal compressive force is partly compensated by the tensile normal force due to the permanent load. To compensate the compressive force acting at the diagonal we can increase the declination of the tie or stress the diagonals artificially with a post-tensional force.

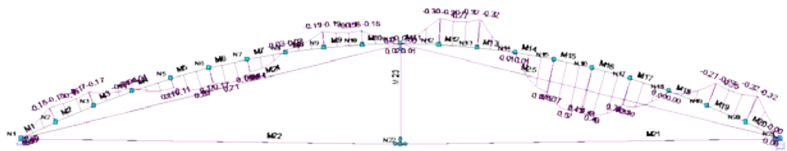

FIGURE 7.6 Bending moments due to the snow load for the vault with an inclined tie to tension the diagonals.

Member	Load:	Nx	Vz	M
4	permanent load	-51.96	1.31	0.57
	live load left	-10.28	0.24	0.14
	snow load	- 6.77	0.29	0.11
	wind over pressure	20.42	0.85	0.78
	wind under pressure	0.53	0.51	1.08
6	permanent load	-49.52	1.41	0.41
	live load left	-9.99	0.31	0.16
	snow load	- 6.53	0.24	0.21
	wind over pressure	20.47	0.69	0.29
	wind under pressure	0.46	0.88	0.71
10	permanent load	-48.11	1.49	0.48
	live load left	- 9.70	0.35	0.07
	snow load	- 6.35	0.21	0.16
	wind over pressure	20.47	0.74	0.37
	wind under pressure	0.49	1.45	1.02
22 tie	permanent load	+45.52		
	live load left	+4.68		
	snow load	+6.98		
	wind over pressure	-14.28		
	wind under pressure	- 5.89		
24, diag-onal	permanent load	+2.62		
	live load left	+5.16		
	snow load	- 0.66		
	wind over pressure	- 4.98		
	wind under pressure	+ 5.94		
25, diag-onal	permanent load	+2.62		
	live load left	- 4.79		
	snow load	+ 5.69		
	wind over pressure	+ 2.05		
	wind under pressure	+ 2.12		

TABLE 7.4 Results, normal forces, shear forces and bending moments for the vault with an inclined tie and vertical strut

Buckling

Due to the permanent load and the windload with underpressure member 4 of the vault , see table 7.4, is subjected to a normal force and bending moment equal to:

$N = 51.96 + 0.53 = 52.5$ kN

$M = 0.57 + 1.08 = 1.65$ kNm

For the serviceability state the stiffness is defined before in chapter 5 with the MNκ-diagram, see figure 5.16, and table 5.6.

For M_d = 1.65 kNm the stiffness is defined by interpolating between respectively:

$M = 1.490 \times 10^6$ Nmm: $\kappa = 0.299 \times 10^{-6}$ mm^{-1}

$M = 2.219 \times 10^6$ Nmm: $\kappa = 0.540 \times 10^{-6}$ mm^{-1}

$$\kappa = 0.299 \times 10^{-6} + (0.54-0.299) \times 10^{-6} \times \frac{(2.219 - 1.65)}{2.219 - 1.490} = 0.487 \times 10^{-6} \text{ mm}^{-1}$$

$$EI = M/\kappa = 1.65 \times 10^6 / (0.487 \times 10^{-6}) = 3.387 \times 10^{12} \text{ Nmm}^2$$

The critical buckling force is calculated for the parabolic vault with a buckling length ½ s = 5.150 m:

$$N_{cr\,asym} = \frac{\pi^2 EI}{s^2} = \frac{\pi^2 \times 3.387 \times 10^{12}}{5150^2} = 1261 \times 10^3 \text{ N}$$

Thus the ratio n_{cr} of the buckling force to the normal force is: $n_{cr} = N_{cr}/N = 1261/52.5 = 24$

Generally a ratio of $n_{cr} \geq 5$ was recommended, thus for the serviceability state the structure meet the demands.

§ 7.3 Ultimate load bearing capacity

Nowadays the reinforcement has to be calculated according to the Eurocode 2. The calculation of the required reinforcement is based on a non-linearly stress-strain diagram of the concrete and steel, as described in chapter 5. The calculation of the resistance of a column or wall is quite labour intensive, so most engineers use diagrams or spreadsheets to calculate the load bearing capacity of a section subjected to an eccentric normal force.

Design loads

For the ultimate state the permanent and live load are increased with a load factor of respectively 1.2 and 1.5.

Due to the permanent load and the windload with underpressure member 4 of the vault , see table 7.4, is subjected to a normal force and bending moment equal to:

$N_d = 1.2 \times 51.96 + 1.5 \times 0.53 = 63.2$ kN

$M_d = 1.2 \times 0.57 + 1.5 \times 1.08 = 2.3$ kNm

Stiffness of the section, ultimate state

As showed before in chapter 5 the stiffness is defined for the given section with $A = 1000 \times 130\ mm^2$ for the ultimate state with the following table and MN-κ diagram. Probably the vault is cracked; for a cracked structure the stiffness can be calculated with expression [5.10]: $EI = M_e/\kappa_e$. Where M_e is the bending moment if the reinforcement reaches the maximum stress. The curvature (kappa) k of a structure loaded by a bending moment and a normal force follows from [5.9]:

$$\kappa = \frac{\sigma_c}{E_c k_x h}$$

Due to the fusees the section of a Fusée Céramique vault is not massive. The compressive zone of a section of a Fusée Céramique vault can be smaller or larger than the compressed flange with a thickness c_f. The bending moment M_e is defined for $k_x < c_f/h$ and $k_x > c_f/h$. Again the curvature and stiffness for the maximum bending bending moment is defined with the procedure showed before in chapter 5.
Features of the section:
Reinforcement: Fe220, E_s = 200000 MPa, f_{sd} = 220/1.15 = 190 MPa,$\omega = A_t/(b\ h)$ = 0.00429, d/h = 0.15; Concrete: C12/15: f_c = 12/1.5 = 8 MPa , E_{cd} = 27000/1.2= 22500 MPa.

According to the Euro-code Young's modulus can de calculated with an effective creep factor ϕ_{ef} following from [5.23]: $\phi_{ef} = \phi_t M_{E\phi}/M_{Ed}$

Due to the permanent load the vault is subjected to a bending moment: $M_{E\phi}$ = 0.57 kNm
The maximum moment due to the asymmetrical load is equal to M_{Ed} = 2.3 kNm. Substituting these moments into [5.23] gives the creep factor ϕ_{ef}:

$$\phi_{ef} = \phi_t M_{rep}/M_{Ed} = 4.0 \times 0.57/2.3 = 0.99$$

For the instantaneous load Young's modulus follows from:

$$E_{ct} = E_{cd}/(1+ \phi_{ef}) = 22500/(1+0.99) = 11306 \text{ MPa}$$

For the permanent and the asymmetrical live load the normal force and bending moment are respectively N_d = 63.2 kN and M_d = 2.3 kNm. The following diagram and table show for the vault subjected to this load the stiffness. The ratio Young's modulus steel/concrete is: $n_s = E_s/E_c = 200 \times 10^3/11306 = 17.7$.

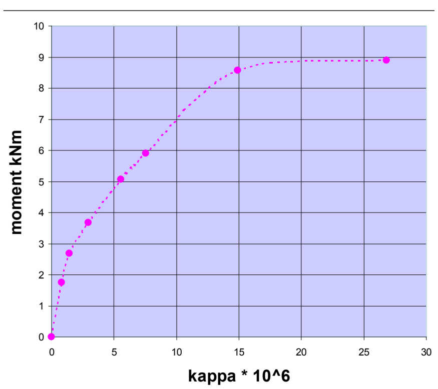

FIGURE 7.7 M-N-κ diagram for N_d=63.2 kN, ultimate state

k_x =	kappa × 10^6 [1/mm]	M [kNm]	EI × 10^{12} [Nmm²]
Infinitive	0	0	
1	0.79	1.76	2.23
0.731	1.44	2.68	1.86
0.5	2.92	3.69	1.26
0.385	5.57	5.06	0.91
0.337	7.56	5.90	0.78
0.269	14.92	8.58	0.58
0.252	26.81	8.90	0.33

TABLE 7.5 Stiffness of the vault, UGT

For M_d = 2.3 kNm the stiffness is defined by interpolating between respectively:

$M_d = 1.76 \times 10^6$ Nmm: $\kappa = 0.79 \times 10^{-6}$ mm^{-1}

$M_d = 2.68 \times 10^6$ Nmm: $\kappa = 1.44 \times 10^{-6}$ mm^{-1}

$$\kappa = 0.79 \times 10^{-6} + (1.44-0.79) \times 10^{-6} \times \frac{(2.3 - 1.76)}{2.68 - 1.76} = 1.17 \times 10^{-6} \text{ mm}^{-1}$$

$$EI_d = M/\kappa = 2.3 \times 10^6 / (1.17 \times 10^{-6}) = 1.97 \times 10^{12} \text{ Nmm}^2$$

With this stiffness the critical buckling force is calculated with expression [3.13] for the parabolic vault and a length from the top to the supports equal to s = 0.5 × 10.31 m.

$$N_{cr\,asym} = \frac{\pi^2 EI_d}{s^2} = \frac{\pi^2 \times 1.97 \times 10^{12}}{5150^2} = 733 \times 10^3 \text{ N}$$

For the asymmeyrical load the normal force is for x = ½ a equal to N_d = 66.7 kN. Thus the ratio n_{cr} of the buckling force and the normal force is equal to: $n_{cr} = N_{cr}/N_d = 733/63.2 = 11.6$

The bending moment inclusive of second order effects is: $M_d = 2.3 \times 11.6/(11.6-1) = 2.5$ kNm.

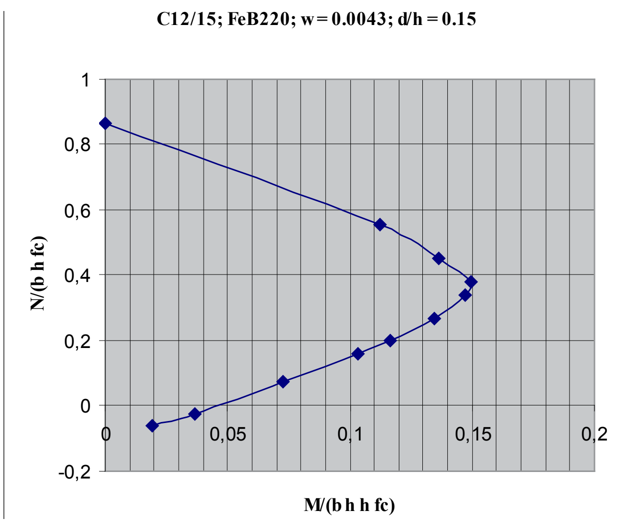

FIGURE 7.8 Graph showing the bearing capacity of the vault for: C12/15; Fe220; d/h = 0.15; ω = 0.0043.

$N_d/(b\,h\,f_{cd})$	$M_d/(b\,h^2\,f_{cd})$
-0.0627	0.0192
-0.0227	0.0368
0.0735	0.0727
0.1592	0.1033
0.2	0.1164
0.2692	0.1347
0.3378	0.1471
0.3804	0.1496
0.4536	0.1362
0.5551	0.1121
0.6396	0.0885

TABLE 7.6 Bearing capacity For C12/15; Fe220; d/h = 0.15; ω = 0.0043

For te vault with C12/15, Fe220 and d/h = 0.15 and a reinforcement $\omega = A_t/(b\,h) = 0.0043$ the bearing capacity is defined with table 7.6 and the graph, see figure 7.8. For the permanent load and the wind load the normal force is for x = ½ a equal to N_d = 63.1 kN. Table 7.6 gives for this normal force the ultimate bending moment, with f_{cd} = 12/1.5 = 8.0 MPa:

$$\frac{N_d}{b\,h\,f_{cd}} = 0.061 \quad \rightarrow \quad \frac{M_d}{b\,h^2 f_{cd}} = 0.069$$

The ultimate moment the section can resist is equal to M_u = 9.3 kNm. This moment is much larger than the calculated bending moment inclusive of second order effects: M = 2.5 kNm. Due to the strengthening the vault can transfer the loads safely.

§ 7.4 Parabolic trussed vault

The structure is strengthened with ties as in a Polonceau truss. To show the distribution of the loads the trussed arch is simplified into a statically determinate truss with straight bars connected by hinges.

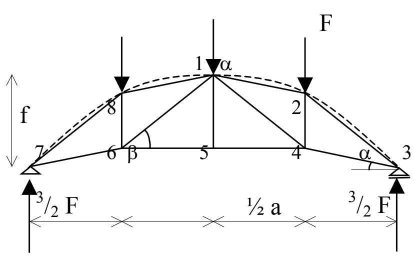

FIGURE 7.9 Trussed arch, subjected to a symmetrically loading

The span and rise of the arch is equal to respectively 2 a = 19.8 m and f = 2.48 m. The length of the strut at the centre is equal to ¾ f = 1.86 m, the length of the two other struts is equal to ½ f = 1.24 m.

The angle between the diagonals and the horizontal is equal to a and b with tan a = ½ f/a = 0.125 and tan b = $^3/_2$ f/a = 0.188 thus cos α = 0.992 and cos β = 0.936. The nodes are counted clockwise.

To simplify the analysis of the forces acting in the ties and struts the structure is assumed to be loaded by concentrated forces acting on the joints. For every node the forces acting at the bars are defined with the vertical and horizontal equilibrium of the forces $\Sigma V = 0$ and $\Sigma H = 0$.

Equally distributed load

The truss is subjected to an equally distributed load, this load subjects the nodes to concentrated forces equal to F, with F= q× a/2, a = 9.9 m and f = 2.48 m the load is equal to: F= 2.4 × 9.9/2 = 11.88 kN. Further cos α = 0.992 and cos β = 0.936 .Due to the permanent load the normal forces acting on the elements are:

S_{12} =	($^3/_2$ F × a/f)/cos α =	- 71.7 kN
S_{23} =	($^3/_2$ F × a/f)/cos β =	- 76.0 kN
S_{34} =	($^3/_2$ F × a/f)/cos α =	+ 71.1 kN
S_{45} =	$^4/_3$ F × a/f =	+ 63.2 kN
S_{14} =	($^1/_6$ F × a/f)/cos β =	+ 8.44 kN

The vault is compressed and all ties are tensioned.

Trussed vault, asymmetrically loaded.

The truss is subjected to an asymmetrically load q = 0.5 kN/m. Due to this load node 1 and 2 are subjected to respectively a load of ½ F and F with F = q a/2. Thus:

F_1 = ½ × 0.5 × 9.9/2 = 1.2375 kN

F_2 = 0.5 × 9.9/2 = 2.475 kN

Due to the asymmetrical live load the normal forces acting on the elements are:

S_{12} = (F × a/f)/ cos α =	- 9.46 kN
S_{23} = (F × a/f)/ cos β =	- 10.55 kN
S_{34} = (F × a/f)/ cos α =	+ 9.96 kN
S_{45} = $^2/_3$ F × a/f =	+ 6.59 kN
S_{14} = ($^1/_3$ F × a/f)/ cos β =	+ 3.52 kN
S_{18} = (½ F × a/f)/ cos α =	- 4.98 kN
S_{78} = (½ F × a/f)/ cos β =	- 5.28 kN
S_{16} = ($^1/_6$ F× a/f)/ cos β =	- 1.76 kN
S_{67} = (½ F × a/f)/ cos α =	+ 4.98 kN

Due to the asymmetrical load the tie S_{16} is compressed: S_{16} = -1.76 kN, but due to the permanent load this tie is tensioned S_{16} = S_{14} = -+8.44 kN. The normal force due to permanent load is larger than the force due to the asymmetric load so this tie is tensioned too. Actually the loads acting at the vault are distributed and not concentrated at the nodes, the vault is thus subjected to bending also. Further the small struts connected with the vault at a quarter of the span are positioned perpendicular to the curve. With a computerprogram, Matrixframe, the forces and bending moments are defined accurately for varying loads.

FIGURE 7.10 Trussed vault, bending moments due to thepermanent load

Member	Load	N_x [kN]	V_z [kN]	M [kNm]
S1	Permanent load	-75.88	2.20	1.14
	Live load right side	- 4.6	0.26	0.26
	Live load left side	-10.88	0.21	0.05
	wind over pressure	25.44	0.75	0.26
	wind under pressure	5.60	1.31	1.12
S4	Permanent load	-72.94	2.45	1.17
	Live load right side	- 4.60	0.32	0.23
	Live load side left	-10.32	0.27	0.08
	wind over pressure	25.43	0.57	0.49
	wind under pressure	5.62	1.04	1.46
S6	Permanent load	-71.38	3.06	2.46
	Live load side right	- 4.57	0.52	0.66
	Live load side left	-10.03	0.38	0.13
	wind over pressure	25.45	1.09	0.39
	wind under pressure	5.50	0.27	1.04
S10	Permanent load	- 69.97	2.91	1.57
	Live load side right	- 4.59	0.34	0.19
	Live load side left	- 9.72	0.26	0.21
	wind over pressure	25.44	1.27	0.62
	wind under pressure	5.56	1.12	0.83
S44	Permanent load	+63.26		
Tie	Live load side right	+ 6.51		
	Live load left side	+ 6.51		
	wind over pressure	-19.84		
	wind under pressure	- 7.99		
S37	Permanent load	7.09		
diagonal	Live load right side	-2.08		
	Live load left side	+3.44		
	wind over pressure	-4.56		
	wind under pressure	3.09		

TABLE 7.7 Trussed vault, normal forces, shear forces and bending moments

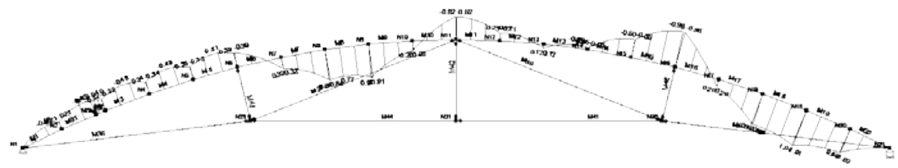

FIGURE 7.11 Trussed vault, bending moments due to the wind over-pressure

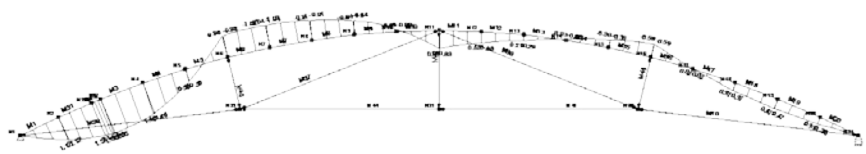

FIGURE 7.12 Trussed vault, bending moments due to the under-pressure

Due to the smaller lever arm the normal forces are about a factor $^{4}/_{3}$. larger than calculated for the vault strengthened with diagonals. Due to the permanent load the vault is subjected to a bending moment equal to M = 2.46 kNm. This bending moment is nearly as large as the bending moment acting in the non strengthened vault, M = 3.1 kNm, due to the live load. The tensile forces acting at ties due to the permanent load always exceed the compressive forces due to the asymmetrical loads and the wind loads. The ties can thus be dimensioned very slender.

§ 7.5 Conclusions

The structure can be strengthened sufficiently with the described methods. Due to the strengthening the bending moments are decreased, the buckling length is halved and the critical buckling force is increased. Strengthening with diagonals is a very effective method. However the diagonals can be subjected to compressive forces. To prevent buckling the diagonals can not be constructed with ties, nevertheless the forces are pretty small so these elements can be quite slender. The strengthening with diagonals is very effective for the renovation of Fusée Céramique vaults if the structure does not meet the demands of the present The joints of the diagonals with the top of the vault are mainly subjected to a compressive force perpendicular to the vault and a shear force parallel to the vault. Alternatives are researched to prevent any compressive normal force acting on the ties. Supporting the top with a vertical strut running to the tie which is inclined downward will introduce an upward force. This upward force will tension the ties.Attaching ties firmly to a Fusée Céramique vault can be difficult. The joints of the ties with the top of the vault are subjected to a tensile force perpendicular to the vault and a shear force parallel to the vault. The resistancy of a bolt constructed into an infill will be poor. To simplify the details the Fusée Céramique vault are by preference strengthened with diagonals capable to resist tensile and compressive forces. Strengthening a vault with a truss is also an interesting alternative. Unfortunately the normal forces and bending moments acting on the elements of the trussed vault exceed the bending moments acting on the vault strengthened with diagonals. Nevertheless due to the statically indeterminacy the trussed vault is a very reliable structure.

8 Re-designing the Fusée Céramique vaults

Generally most roofs are designed to resist the relatively modest loads caused by wind, snow, rain and so on. Possibly it would be better to increase the resistance of roofs so these roofs can be used for example leisure or to produce food and solar energy. Green roofs can contribute much to the ecosystem and climate. A green roof can retain water to reduce the capacity of the sewing system, decrease the extreme temperatures in an urban area and help to clean the atmosphere from pollution. For these roofs the live and permanent loads are much higher than the loads due to wind, rain or snow, so these roofs must be designed stronger and stiffer than for conventional roofs. The cost of construction will rise but the benefits will be significant. In the past the Fusée Céramique vaults were constructed very slender, especially due to the ceramic infill elements the self-weight and the need for cement were minimal. Nowadays it is again important to minimise the need for cement to reduce the embodied energy and CO_2 emission of buildings. Form-active structures can transfer very efficiently with a minimum thickness and reduced need for material the loads for green roofs, accessible for public, or the loads due to installations, machines and equipment to produce food and energy. In the past the Fusée Céramique system was competitive. Nowadays this system has to be updated and redesigned to meet present day demands. This chapter focuses on the redesign of these form-active concrete vaults capable of transferring substantial loads, so these roofs are useful for all kind of activities. Especially the construction method and the selection of the infill elements will be explored to reduce the environmental impact of these vaults.

§ 8.1 Environmental load of floors

Nowadays roofs are designed to resist a modest live load. To create roofs useful for leisure, production of energy and food, the load bearing capacity of roofs has to be increased, so these roofs can transfer a live load comparable to the live load acting on a floor.

Type	Thickness [mm]	Class	Mass [kg/m²]	Embodied energy [MJ /m²]	CO_2 [kg/m²]
Concrete slab made in situ	170	7a	430	1272	99.3
Wide slab floor including concrete finishing and 16 kg/m² reinforcement	50 + 120	3c	424	1207	99.9
Prefabricated rib-cassette elements	250	6a	220	796	58.0
Ceramic elements supported by concrete beams, with a concrete finishing	150 + 40	6c	274	889	64.8
Prefabricated hollow core elements	150	6a	252	625	52.3
Miltiplex supported by timber joists	45 × 200	3b	22	22	42,8
Multiplex supported by timber joists, sustainable production	45 × 200	1a	22	22	-14,3
Prrefabricated gas concrete elements		6a	150	732	63.9
PS elements supported by prefab. concrete beams	200	5c	205	636	47.3

TABLE 8.1 Mass , embodied energy and CO_2 emission for floors with a span of 7.2 m [Nib12]

The research institute Nibe classified the environmental load for several floor systems. Table 8.1 shows for some floor systems with a span of 7.2 m the classification concerning the environmental load according to this institute [Nib12].
In practice concrete floors are used widely, nevertheless the environmental load of a massive concrete floor slab is substantial. The weight, embodied energy and CO_2 emission of prefabricated rib-cassette and prefabricated hollow core elements is about the half of the embodied energy and emission of a concrete slab made in situ. The minimum weight, embodied energy and emission are found for multiplex floors supported by timber joists.
Increasing the span will increase the mass and environmental impact non-proportionally. Generally the span of a roof on a workshop, factory, warehouse or sporting halls is larger than the span of a floor in a office building, which generally does not exceed 10-12 m. For the design of a roof with a long span reducing the self-weight is quite important. Floors are constructed with a horizontal surface, but roofs are constructed with a curvature or inclination to drain rainwater and snow. Due to the curvature form-active structures are very efficient. For these structures the need of material and self-weight is minimal. For the barrel vault of building Q in Woerden, with a span of 19.8 m, the thickness versus the span ratio was only 130/19800 = 1/152. Due to the ceramic infill elements the need for cement and the self-weight was reduced respectively 27% and 41%. Consequently the emission of CO_2 and the embodied energy was reduced as well. Possible the footprint and environmental impact can be reduced further. To select infill elements the features of varying materials concerning environmental load, production, construction, form, strength and stiffness are researched.

§ 8.2 Uplift infill elements during construction

Light infill elements float upward during construction, if the weight of the elements is less than the upward load due to the density difference. For a fusée Ø80-60 with a mass of m_{fusee} = 1800 kg/m³ and a length l = 1.0 mthe weight is equal to:

$$W = \pi/4 \times (80^2 - 60^2) \times 18 \times 10^{-6} = 0.04 \text{ kN}$$

According to Archimedes the upward force F_{up} caused by the liquid concrete is equal to the volume times the mass of the liquid concrete. For a length l = 1.0 m the upward force in liquid concrete with a density of m_m = 2400 kg/m³ is equal to:

$$F_{up} = \pi m_m r^2 = \pi \times 24 \times 0.04^2 = 0.12 \text{ kN}$$

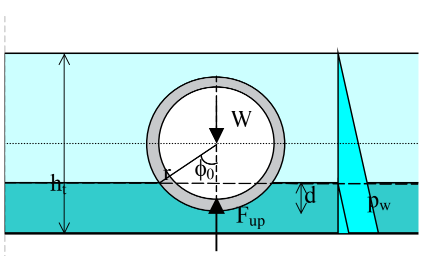

FIGURE 8.1 Uplift of the infill element pushed in the first layer

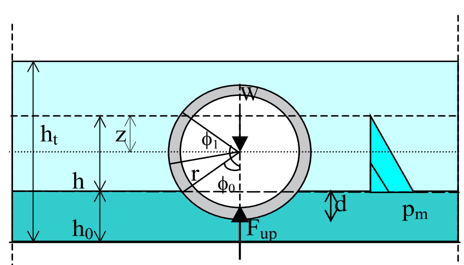

FIGURE 8.2 Uplift acting on the infill while the second layer is poured

The elements must be fixed firmly to the mould or must be ballasted if during the construction the upward force F_{up} is larger than the weight W. For the Fusée Céramique vaults pouring the liquid concrete in two layers solved this problem. Firstly a thin layer of liquid concrete of about 2.5 cm was poured, the fusées were pushed into the liquid concrete to a depth of about 1 cm. Next the second layer of liquid concrete was poured onto the semi-cured first layer. The elements have to be pushed into the liquid concrete to prevent the uplift with a dept d. This depth can be described with polar coordinates:

$$d = r\,[1 - \cos\phi_0] \qquad [8.1]$$

The upward force acting on the element follows *per* unit length *l* follows from:

$$F_{up} = m_m \times [\textstyle\int_0^{\phi 0} 2 \times {}^1/_2 \times r^2\, d\phi - 2 \times {}^1/_2 \times r^2 \times \sin\phi \times \cos\phi\,] \quad \rightarrow$$

$$F_{up} = m_m\, r^2\, [\phi_0 - {}^1/_2 \sin(2\,\phi_0)\,] \qquad [8.2]$$

Example

A fusée with radius r = 40 mm, length l =1.0 m is pushed into the liquid concrete with m_m = 24 kN/m^3. Assume $\phi_0 = ¼\,\pi$ radians, the depth d is equal to: $d = r \times [1 - \cos\phi_0] = 12$ mm
The upward force *per* unit length *l* is equal to:

$$F_{up} = 24 \times 0.04^2 \times [¼\,\pi - ½ \sin(½\,\pi)] = 0.011 \text{ kN}$$

The upward force acting on the fusées is smaller than the weight W = 0.04 kN, thus the infill does not float.

Upward force during the construction of a second layer

Next the upward force is calculated during the construction of the second layer if the liquid concrete rises to a certain level h above the first layer h_0. The height of the liquid concrete h varies during the construction and is at maximum equal to the depth of the structure h_t minus the depth h_0 of the first layer poured previously. Assume the cover on the infill element at the upper side is equal to c. During the construction the liquid concrete rises slowly till the height h of the liquid concrete is equal to $h_{max} = h_t - h_0$. For a circular infill element with a radius r the height h with $h < h_t - c - h_0$ is described with:

$$h = r\,(\cos\phi_0 - \cos\phi_1) \qquad [8.3]$$

The pressure of the liquid concrete acting on the infill depends on the depth z below the surface temporary at a height h_1. The pressure of the liquid concrete is equal to: $p_z = m_m z$, with z follows from:

$$z = r\,(\cos\phi - \cos\phi_1) \qquad [8.4]$$

The radial force acting on a small part of the element *per* unit length *l* with a width r dϕ is equal to:

$$dF_\phi = m_m\, z\, r\, d\phi$$

The vertical component of this force is equal to: $dF_{\phi v} = dF_\phi \cos\phi$, with $\phi_0 \le \phi \le \phi_1$.

The upward force acting at the infill is calculated by integration of $dF_{\phi v}$ over the interval $\phi_0 \le \phi \le \phi_1$:

$$F_{up} = 2 \int dF_{\phi v} = 2\, r^2\, m_m \int_{\phi 0}^{\phi 1} [\cos^2\phi - \cos\phi_1 \times \cos\phi]\, d\phi$$

Integrating this expression between $\phi = \phi_0$ and $\phi = \phi_1$:

$$F_{up} = r^2\ m_m\ [-\sin\phi_1 \times \cos\phi_1 - \sin\phi_0 \times \cos\phi_0 + \phi_1 - \phi_0 + 2 \times \sin\phi_0 \times \cos\phi_1] \quad [8.5]$$

Table 8.2 shows for a liquid concrete layer, raised till a height h, the upward force acting at the infill with radius r = 40 mm. Assume the angle ϕ_0 is equal to $\phi_0 = ¼\,\pi$. Then the infill is pushed into the first layer over a depth:

$$d = r\ [1 - \cos(¼\,\pi)] = 12\ \text{mm}.$$

The height of the liquid concrete varies during construction and is described for $\phi_0 = ¼\,\pi$ with [8.3].

$$h = r\,(\cos(¼\,\pi) - \cos\phi_1)$$

The following table shows that if the upward force is less than the weight, the infill will not float.

ϕ_1 =	height: $h/r = \cos(¼\,\pi) - \cos\phi_1$	h [mm]	$F_{UP} / (l\ r^2\ m_m)$	F_{UP} / l [kN/m]
½ π	$h/r = \cos(¼\,\pi) - \cos(½\,\pi) = 0.707$	28	0.285	0.011
¾ π	$h/r = \cos(^1/_4\,\pi) - \cos(^3/_4\,\pi) = 1.414$	57	0.571	0.022
π	$h/r = \cos(¼\,\pi) - \cos(\pi) = 1.707$	68	0,442	0.017

TABLE 8.2 Uplift force F_{up} acting on an infill in a liquid concrete with varying rise for r = 0.04 m, a length l = 1.0 m and $\phi_0 = ¼\,\pi$

Fixing infill elements

Casting the concrete in several layers is time consuming and increases the cost. The concrete can also be poured in one single stage, provided the uplift force acting at the infill is resisted. For a bridge over the canal Buinen-schoonoord in the Netherlands the infill tubes were fixed with a steel frame that was bolted to the mould [Rij59]. For a curved prefabricated element, the reinforcement, following the curve of the element and connected firmly to the infill, can resist the upward load acting on the infill elements easily, this will be showed with the following example.

Example

Assume a roof is composed of prefabricated elements. The length of the prefabricated element is a = 8.0 m and the rise of the prefabricated element is equal to f = a/16 = 0.5 m. The centre-to-centre distance of the infill elements with a diameter of 80 mm is 90 mm. the elements are reinforced with Ø6-180 at both sides. For a width of 1.0 m the upward load acting on the infill elements is:

$$q = 11 \times \pi \times r^2 \times 24 = 11 \times \pi \times 0.04^2 \times 24 = 1.33\ \text{kN/m}$$

For a prefabricated element with a length a = 8 m and f =0.5 m, a width b = 1.0 m the tensile force is equal to:

$$F = {}^1/_8\, q\, a^2/f \quad \rightarrow \quad F = {}^1/_8 \times 1.33 \times 8^2/0.5 = 21.3\ \text{kN}$$

During the construction the force has to be resisted by the rebars 2Ø6-180. The stress is equal to:

$$\sigma = F/A_s = 21300/(2\times157) = 68\ \text{MPa} < f_s$$

The stress is smaller than the ultimate stress f_s for the reinforcement.

Cassettes

The liquid concrete will not push the infill elements upward if these elements are connected well to the bottom of the mould and the liquid concrete cannot flow between the infill and bottom of the mould. The infill elements can be a part of the mould or made of prefabricated cassettes with a brim fitting to the mould. The section of the vault can be considered as composed of T-elements, see figure 8.3.

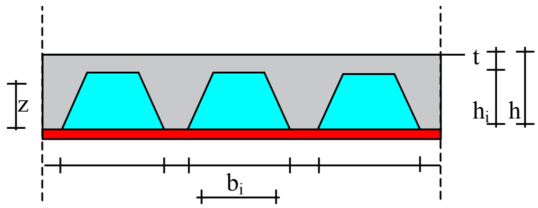

FIGURE 8.3 Cassette infill elements, connected to the bottom of the mould, to prevent uplift.

§ 8.3 The strength of the tubular infill elements

During the construction the infill elements have to resist the loads when the liquid concrete is poured into the mould. For an infill with the centre at a depth z the radial load acting at the surface is:

$$p_{\phi 1} = m_m\ z + m_m\ r\ \cos\phi_1$$

Due to the first part of this equation the element is subjected to a normal compressive force and due to the second part of the equation the element is subjected to bending.

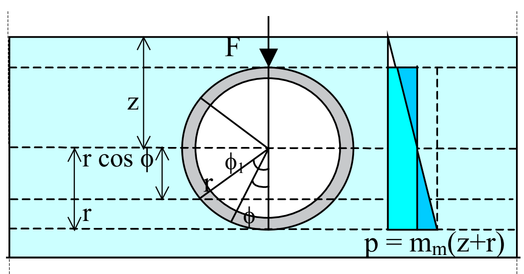

FIGURE 8.4 Loads acting on an infill element during the construction.

Firstly the load is analysed as if the element is fixed to the mould at the top, see figure 8.4. To describe the bending moments the element is split into two equal parts. At the top both parts are clamped. At the bottom of the element the bending moment is supposed to be $M_{\phi 0}$. The bending moment for an angle $\phi = \phi_1$ follows from:

$$M_{\phi 1} = M_{\phi 0} - \int (m_m\, r \cos\phi \times r\, d\phi) \times r \sin(\phi_1 - \phi)\, d\phi$$

Substituting $\sin(\phi_1 - \phi) = \sin\phi_1 \times \cos\phi - \cos\phi_1 \times \sin\phi$ gives:

$$M_{\phi 1} = M_{\phi 0} - m_m\, r^2 \times \int_{\phi=0}^{\phi 1} [\sin\phi_1 \times \cos^2\phi - \cos\phi_1 \times \sin\phi \times \cos\phi]\, r\, d\phi$$

Integrating between $\phi = 0$ and $\phi = \phi_1$ gives: $M_{\phi 1} = M_{\phi 0} - \frac{1}{2} m_m r^3 \sin\phi_1 \times \phi_1$
[8.7]

For an angle $\phi = 0$ the rotation is zero and $M_{\phi 1} = M_{\phi 0}$. For an angle $\phi > 0$ the rotation follows from:

$$\frac{\int_{\phi=0}^{\phi} M_\phi \, r \, d\phi}{EI} - \frac{\int_{\phi=0}^{\phi} \frac{1}{2} m_m r^3 \sin\phi \times \phi \times r \, d\phi}{EI}$$

For an angle $\phi = \pi$ the rotation is zero. Integrating between $\phi = 0$ and $\phi = \pi$ gives:

$$\frac{\int_{\phi=0}^{\phi=\pi} M_\phi \, r \, d\phi}{EI} - \frac{\int_{\phi=0}^{\phi=\pi} \frac{1}{2} m_m r^4 \times (\sin\phi \times \phi) \, d\phi}{EI} = 0$$

$$\frac{M_{\phi 0} \, r \, \pi}{EI} = \frac{\frac{1}{2} m_m r^4 \times [-\phi\cos\phi + \sin\phi]_0^{\pi}}{EI} \quad \rightarrow \quad M_{\phi 0} = \frac{1}{2} m_m r^3$$

Substituting $M_{\phi o}$ in [8.7] gives: $M_\phi = \frac{1}{2} m_m r^3 \times [1 - \sin\phi \times \phi]$ [8.8]

The bending moments are at maximum for $\phi = 0$ and $\phi = \pi$: $M_{\phi=0} = \frac{1}{2} m_m r^3$ and $M_{\phi=\pi} = \frac{1}{2} m_m r^3$
For a fusée with a radius of r=0.04 m the bending moment is:

$$M_{\phi=0} = M_{\phi=\pi} = \frac{1}{2} m_m r^3 = \frac{1}{2} \times 24 \times 0.04^3 = 0.000768 \text{ kNm/m}$$

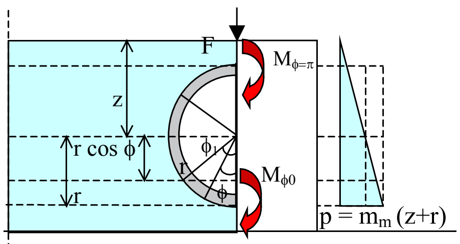

FIGURE 8.5 Loads acting on infill element fixed to the mould at the top.

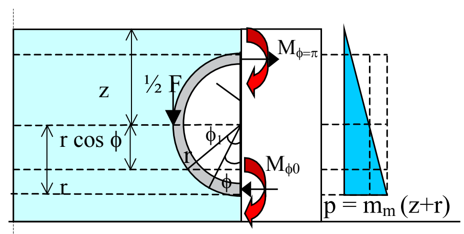

FIGURE 8.6 : Loads acting on infill element during construction the infill is fixed to the mould at the side.

The stress acting in a fusee element follows from $\sigma = M/W$. For a tube with thickness 10 mm the modulus of the section is $W = 1000 \times 10^2/6$ mm^3, then the bending stress is equal to:

$$\sigma = 0.5 \times 10^{-3} \text{ MPa}$$

The upward force follows from: $F = 2 m_m r^2 \times \int_{\phi=0}^{\phi 1} \cos^2\phi \, d\phi$

Integrating between $\phi = 0$ and π gives: $F = 2 \times \frac{1}{2} \pi m r^2$

Due to this load the fusée is subjected to a normal force and normal stress:

$$\sigma = \frac{1}{2} F/(t\,l) = \frac{1}{2}\pi \times 24 \times 0.04^2 \times 10^3/(10 \times 1000) = 6 \times 10^{-3} \text{ MPa}$$

The calculated stresses are quite low, to resist the normal force and bending moments the fusée elements can be made thinner.

Next the load is analysed as if the element is fixed to the mould at the sides, see figure 8.6. To describe the bending moments the element is split into two equal parts. At the top both parts are clamped. At the bottom of the element the bending moment is supposed to be $M_{\phi 0}$. Due to the eccentric load the element is subjected to horizontal forces H. These horizontal forces follow from the equilibrium of bending moments:

$$2\,r\,H = {}^1/_2\,F\,r - (M_o + M_\pi), \qquad \rightarrow \qquad H = {}^1/_4\,F - ½\,(M_o + M_\pi)/r$$

Substituting ${}^1/_2\,F = ½\,\pi\,m\,r^2$ gives: $\qquad H = ¼\,\pi\,m\,r^2 - ½\,(M_o + M_\pi)/r$ [8.9]

The bending moment for an angle $\phi = \phi_1$ follows from:

$$M_{\phi 1} = M_0 + H\,r\,(1-\cos\phi_1) - \int_{\phi=0}^{\phi 1} m_m\,r\cos\phi \times r\sin(\phi_1 - \phi)\,r\,d\phi$$

Substituting H and $\sin(\phi_1 - \phi) = \sin\phi_1\cos\phi - \cos\phi_1\sin\phi$ gives:

$$M_{\phi 1} = ½M_0\,(1+\cos\phi_1) - ½M_\pi\,(1-\cos\phi_1) + ¼\,\pi\,m\,r^3(1-\cos\phi_1) - m_m r^3 \int_{\phi=0}^{\phi 1}[\sin\phi_1\cos^2\phi - \cos\phi_1\sin\phi\cos\phi]\,d\phi$$

Integrating between $\phi = 0$ and $\phi = \phi_1$ gives:

$$M_{\phi 1} = ½\,M_0\,(1 + \cos\phi_1) - ½\,M_\pi\,(1 - \cos\phi_1) + ¼\,\pi\,m\,r^3(1 - \cos\phi_1) - ½\,m_m r^3\,\phi_1\sin\phi_1$$

For $\phi_1 = \phi$:

$$M_\phi = ½\,M_0\,(1+\cos\phi) - ½\,M_\pi\,(1-\cos\phi) + ¼\,\pi\,m\,r^3(1-\cos\phi) - ½\,m_m r^3\,\phi\sin\phi \qquad [8.10]$$

Due to the bending moments the surface of the tube rotates. For an angle $\phi = 0$ the rotation is zero and $M_{\phi 1} = M_{\phi 0}$. For an angle ϕ_1 the rotation follows from: $\theta = \int_{\phi=0}^{\phi 1} \frac{M_{\phi 1}\,r\,d\phi}{EI}$

$$\theta = \int_{\phi=0}^{\phi 1} \frac{M_0\,(1 + \cos\phi)\,r\,d\phi}{2\,EI} - \int_{\phi=0}^{\phi 1} \frac{M_\pi\,(1 - \cos\phi)\,r\,d\phi}{2\,EI} + m_m r^4 \int_{\phi=0}^{\phi 1} \frac{½\,\pi\,(1 - \cos\phi) - \phi\sin\phi\,d\phi}{2\,EI}$$

Integrating between $\phi = 0$ and $\phi = \phi_1$ gives:

$$\theta = \frac{M_0\,(\phi_1 + \sin\phi_1)\,r}{2\,EI} - \frac{M_p\,(\phi_1 - \sin\phi_1)\,r}{2\,EI} + \frac{m_m r^4[½\,\pi\,(\phi_1 - \sin\phi_1) - (-\phi_1\cos\phi_1 + \sin\phi_1)]}{2\,EI}$$

For $\phi = \pi$ the rotation is zero:

$$0 = \frac{M_0\,(\pi + 0)\,r}{2\,EI} - \frac{M_\pi\,(\pi - 0)\,r}{2\,EI} + \frac{m_m r^4[½\,\pi\,(\pi - 0) - (\pi + 0)]}{2\,EI} \quad \rightarrow \quad M_0 = M_\pi - m_m r^3\,({}^1/_2\,\pi - 1)$$

Substituting M_0 into expression [8.10] gives:

$$M_\phi = ½\,[M_\pi - m_m r^3({}^1/_2\pi - 1)] \times (1 + \cos\phi) - ½\,M_\pi\,(1 - \cos\phi) + ½\,m_m r^3[{}^1/_2\pi\,(1 - \cos\phi) - \phi\sin\phi]$$

$$M_\phi = M_\pi\cos\phi + {}^1/_2\,m_m r^3[-\pi\cos\phi + 1 + \cos\phi - \phi\sin\phi]$$

For $\phi = \pi$ the bending moment is equal to:

$$M_\pi = -M_\pi + {}^1/_2\pi\,m_m r^3[\pi + 1 - 1 - \pi \times 0] \quad \rightarrow \quad M_\pi = ¼\,\pi\,m_m r^3 = 0.785\ m_m r^3$$

Next this bending moment M_p is substituted the expression for M_ϕ for $\phi = 0$:

$$M_0 = ¼\,\pi\,m_m r^3 + ½\,m_m r^3(-\pi + 1 + 1) \quad \rightarrow \quad M_0 = ½\,m_m r^3[2 - ½\pi] = 0.2146 \times m_m r^3$$

For a fusée with a radius of 40 mm the bending moment is:

$$M_{\phi=\pi} = ¼\,\pi\,m\,r^3 = ¼\,\pi \times 24 \times 0.04^3 = 0.0012\ \text{kNm/m}$$

For a tube with thickness 10 mm, the modulus of the section is equal to: $W = 1000 \times 10^2/6\ \text{mm}^3$;
The stress acting in a fusee element follows from: $\sigma = M/W \quad \rightarrow \quad \sigma = 0.00072\ \text{MPa}$

This stress is very small, to resist the bending stresses due to the pressure of the liquid concrete the fuse elements can be dimensioned thinner.

§ 8.4 The length and curvature of the oblong infill elements

To follow the curvature of the vault infill elements, positioned parallel to the curvature, must be bent. A tube can be curved elastically if the diameter is limited and the stress due to the bending does not exceed the maximum stresses. The ratio of diameter versus length of the infill elements depends on the features of the chosen material.
The deformation of an element subjected to an equally distributed load is equal to:

$$\Delta = \frac{5 q l^4}{384 EI} \quad [8.11]$$

For an element supported at the ends the maximum bending moment is equal to $M = q l^2/8$. Due to this bending moment the stress is at maximum $\sigma = M/W$, W is the modulus of the section equal to $W = I/z$. For a symmetrical section with a thickness t the distance z from the centre to the upper of lower side is ½ t , thus $I = W \times t/2$. Substituting the bending moment, stress and modulus of the section into the expression of the deformation gives:

$$\Delta = \frac{5 \sigma l^2}{24 E t} \quad [8.12]$$

Generally for a Fusée Céramique vault the ratio rise to the span was about $f/l = {}^1/_8$. An infill with a length equal to the length of the vault must be curved so that the ratio of the deformation to the span is equal to $\Delta/l = f/l = {}^1/_8$. Substituting this ratio into the expression for the elastic deformation gives the following expression for the ratio of thickness to span:

$$t/l = \frac{5 \sigma}{3 E} \quad [8.13]$$

For steel S235 with a maximum stress of 235 MPa and a Young's modulus of 2.1×10^5 MPa the ratio thickness/span is 1/536 = 0.0187. For a span of 8 m the thickness of the infill is 14.9 mm.
For timber with a maximum stress of 10 N/mm^2 and Young's modulus of 10^4 MPa the ratio thickness/span is 1/600 = 0.0167. For a span of 8 m the thickness of the infill is 13.3 mm.

The diameter of an infill that is to be curved elastically has to be quite small. A practical solution to overcome this problem is composing the infill of several tubes, for example by composing the infill elements with an organic material as bamboo, bundled in tubular shoves. The infill elements can be made also of small timber elements curved one by one and glued together. Elements can be curved also non-linearly. For example steel tubes can be forged, bamboo elements and PVC tubes can be curved by heating. The infill can be composed also of small straight elements jointed together as a chain. The Fusée Céramique elements were made as cylindrical bottles with a conical end so the top of an element could be pushed into the rear of the next element. The length of the individual parts has to be low to allow them to follow the curvature of the vault. The maximum length of the elements is defined by the curvature and the tolerance Δt.
For a circle segment the radius R is equal to: $R = ½ (a^2+f^2)/f$ [8.14]

The facetted elements follow the chords of the circle. For an element with length *l* and a tolerance Δt the radius is:

$$R = ½ (^1/_4\, l^2 + \Delta t^2)/\Delta t \qquad [8.15]$$

For a circular segment the radius is constant. With [8.14] and [8.15] the length of the element follows from:

$$(^1/_4\, l^2 + \Delta t^2) = (a^2+f^2)\,\Delta t\,/f \quad \rightarrow \quad l = 2 \times [\,(a^2+f^2)\,\Delta t/f - \Delta t^2\,]^{1/2}$$

The tolerance is much lower than the span and the rise, neglecting Δt^2 results in:

$$l = 2\ a\ [\,(1 + f^2/a^2)\,\Delta t/f\ ]^{1/2} \qquad [8.16]$$

Example

For a vault with span 2.a = 16 m, a rise f = 2.0 m and a tolerance of 5 mm the maximum length of a facetted infill element is: $l\ = 2 \times 8.0 \times [\,(1 + (¼)^2) \times 0.005/2.0\,]^{1/2} = 0.82$ m

§ 8.5 Form, strength and stiffness

The shape and position of the infill elements affects the area, the second moment of the area and the modulus of the section. Consequently the structural efficiency of the vault is also affected. Three alternatives are compared, a massive section, a section with tubular infill elements and a TT-section constructed with boxes.

For a massive section, a height h and a width b the area A and second moment of the area I follows from:

$$A = b\,h \qquad [8.17]$$

$$I = {}^1/_{12}\, b\,h^3 \qquad [8.18]$$

For a section, with a height h, a width b and a number of n tubular infill elements with diameter d_i the area A and second moment of the area I follows from:

$$A = b\,h\ - n\,\pi\,d_i^2 \qquad [8.19]$$

$$I = {}^1/_{12}\, b\,h^3 - n\ \pi\,d_i^2 \qquad [8.20]$$

For a multiple T-section, with a height h, a width b and a number of n infill elements with height h_i and width b_i the area A, the centre of gravity z and second moment of the area I follows from:

$$A = b\,h\ - n\,b_i\,h_i \qquad [8.21]$$

$$z = \frac{½\,b\,h^2\ - n \times ½\,b_i\,h_i^2}{b\,h\ - n\,b_i\ h_i} \qquad [8.22]$$

$$I = b\,h\,[^1/_{12}\,h^2 + (z - ½\,h)^2\,] - n\,b_i\,h_i\,[^1/_{12}\,h_i^2\ + (z - ½\,h)^2] \qquad [8.23]$$

Section	height [mm]	infill height [mm]	infills width [mm]	Center of gravity z [mm]	Area [mm²]	Second moment of the Area [mm⁴]	Modulus of the section [mm³]	dead load [kN/m²]
	h	h_1	b_1					
Massive	110	-	-	½ × 110	11.0×10^4	1.109×10^8	2.02×10^6	2.64
11 tubular infills	110	60		½ × 110	7.9×10^4	1.039×10^8	1.89×10^6	1.90
TT-section	110	60	400	74	6.2×10^4	0.433×10^8	0.59×10^6	1.49
TT-section	180	130	400	124	7.6×10^4	1.86×10^8	1.49×10^6	1.82
4×4 balls - 250	250	180	180	125	14.8×10^4	10.96×10^8	8.76×10^6	4.83

TABLE 8.3 Area and second moment of the area for several sections with a width b = 1.0 m and length l = 1.0 m

Table 8.3 shows that tubular infill elements reduce the area and reduce the second moment of the area and the modulus of the section only slightly.
Next a TT-section is considered, see figure 8.3. The volume of the infill elements can be increased if the elements are a part of the mould and not surrounded by the liquid concrete. Attaching the infill elements to the bottom of the mould will prevent uplift by the liquid concrete so the liquid concrete does not have to be poured in two layers. This will speed up the production. By preference the infill elements are a part of the mould and reused several times. The infill elements are removed when the mould is separated from the concrete. Reusing the infill elements several times will reduce the environmental load of the infill elements. Further the specific surface of the roof is increased, this will increase the accumulation of heat and cold in the roof structure and reduce the energy to warm and cool down the interior. Architecturally the surface is not smooth, a grid of small cassettes is shown.
Using rectangular infills reduces the need for cement and the weight, but to fill the space between the boxes properly the spacing has to be enlarged. For an element with a height of 110 mm and two boxes with a height of 60 mm and a width of 400 mm the volume is 56% of the volume of a massive vault, but for this TT-section the second area of the section and the modulus of the section decrease quite much. To increase the stiffness and strength the height of the TT section is increased to 180 mm. For this element with a height of 180 mm and boxes with a height of 130 mm and a width 400 mm the volume is 69% of the volume of a massive vault.
Concerning the production and assembly it is much easier for cilindrical vaults to position the elements perpendicular to the span, the consequences will be shown in the following paragraph. In the same way it would be easier to use spherical balls for double curved forms.
In practice floors are made composed of a prefabricated wide slab plate with spherical infill elements and a top layer cast in situ. The depth of these floors varies from 250 to 400 mm. The reduction of the weight is about 20%. For a floor with sperical infill elements Ø180, center-to-center 250 mm, the deph will be at least 250 mm. The wide slab plates are jointed with the top layer, the joints can be considered as fixed joints. Due to the fixed joints the structure can be composed of several wide slab plates and can be schematised as a two hinged vault.

§ 8.6 Embodied energy and footprint

In the past Jaques Couëlle was inspired by bamboo, possibly an infill of a natural material such as bamboo can perform better than a ceramic element. Table 8.4 shows for varying materials the mass, embodied energy and emission of CO_2

Product	Density [kg/m³]	Energy [MJ/kg]	CO_2 burden [kg/kg]	
Low carbon steel	7800 – 7900	30.0 – 35.0	2.2 – 2.8	[Ash12]
Brick	1900 – 2100	2.2 – 3.5	0.2 – 0.23	[Ash12]
Concrete	2200 – 2600	1.0 – 1.3	0.13– 0.15	[Ash12]
Soda lime glass	2440 – 2490	14.0 –17.0	0.7 – 1.0	[Ash12]
Rigid polymer foam LD	36 – 70	105– 110	3.5 – 4.0	[Ash12]
Bamboo	600 – 800	4.0 – 6.0	0.3 – 0.33	[Ash12]
Wood	600 – 800	7.0 – 8.0	0.4 – 0.46	[Ash12]
Cardboard	600 - 800	24.8	1.32	[Ham08]

TABLE 8.4 Overview of CO_2 burden and embodied energy for some materials

Comparing brick and concrete shows that brick has a higher embodied energy than concrete but a slightly lower CO_2 emission. However for a hollow element the mass and embodied energy will be much lower than for a massive element, consequently the mass and embodied energy of a roof construction is decreased by using ceramic elements. The embodied energy for bamboo and wood according to Ashby seems rather high. Nibe gives for timber floors made in the Netherlands an embodied energy of about 1.0 MJ/kg. Cardboard tubes can also be an interesting alternative. For example cardboard is composed of about 95% recycled material. Nevertheless cardboard is not used much for structures, mostly because the strength and stiffness are pretty low. Halfway the twentieth century infills cardboard tubes were embedded in concrete floors to reduce the weight and use of cement. Possibly cardboard tubes can reduce the environmental burden of vaults. Ashby does not describe this material, the embodied energy and CO_2 burden for cardboard is adopted from the inventory described by Hammond et al [Ham08].

For a vault with a thickness of 110 mm the weight, embodied energy and CO_2 emission is defined for varying infills. Table 8.5 shows the results for this vault. Figure 8.7 compares the weight, embodied energy and CO_2 burden for vaults with varying infills with a massive vault.

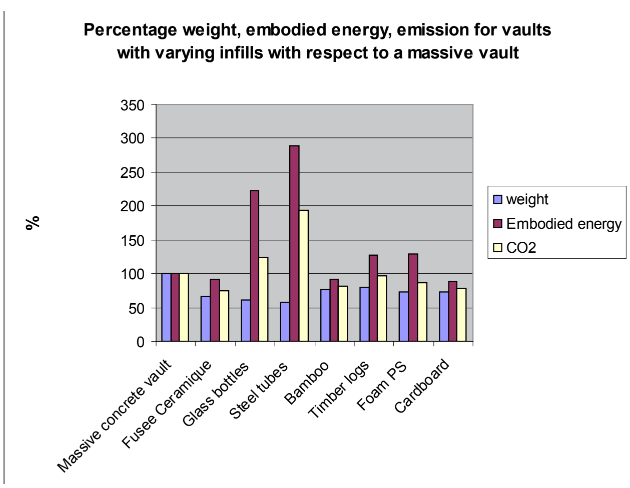

FIGURE 8.7 The weight, embodied energy and CO_2 emission for vaults with varying infill with respect to a massive vault.

Product	mass	Volume	weight	Embodied Energy		CO_2	
	kg/dm³	m³	kg	MJ/kg	MJ/m²	kg/kg	kg/kg
Massive concrete vault	2.4	0.1100	**264**	1.1	290.4	0.14	**37.0**
Fusée Céramique Ø80-10	1.8	0.0242	43.5	2.8	121.9	0.21	9.1
Concrete	2.4	0.0547	131.3	1.1	144.4	0.14	18.4
Concrete + fusée Ø80-10			**174.8**		**266.4**		**27.5**
Glass Ø80-10	2.5	0.0130	32.4	15.5	502.2	0.85	27.5
Concrete	2.4	0.0547	131.3	1.1	144.4	0.14	18.4
Concrete + glass Ø80-10			**163.2**		**639.6**		**45.9**
steel tubes Ø80-1	7.85	0.0030	21.4	32.5	696.5	2.50	53.6
Concrete	2.4	0.0547	131.3	1.1	144.4	0.14	18.4
Concrete +steel tubes Ø80-1			**152.7**		**840.9**		**72.0**
Bamboo Ø60-10	0.7	0.0173	12.1	5.0	60.5	0.31	3.7
Concrete	2.4	0.0789	189.4	1.1	208.3	0.14	26.5
Concrete + bamboo Ø60-10			**201.5**		**268.8**		**30.2**
Timber, Ø60	0.7	0.0310	21.7	7.5	163.3	0.43	9.4
Concrete	2.4	0.0789	189.4	1.1	208.3	0.14	26.5
Concrete + timber Ø60			**211.1**		**371.6**		**35.9**
rigid polymer foam Ø60	0.05	0.0311	1.6	107.5	167.2	3.70	5.8
Concrete	2.4	0.0789	189.4	1.1	208.3	0.14	26.5
Concrete + polymer foam Ø60			**134.2**		**375.5**		**32.3**
Cardboard Ø60-1.4	0.7	0.0028	2.0	24.8	49.2	1.32	2.6
Concrete	2.4	0.0789	189.4	1.1	208.3	0.14	26.5
Concrete + cardboard Ø60-1.4			**191.4**		**257.5**		**29.1**

TABLE 8.5 The weight, CO_2 burden and embodied energy for varying infills embedded in a vault with a depth of 110 mm.

The infills can be ranked with respect to the weight, embodied energy or CO_2 burden. Table 8.6 shows a ranking of the infills concerning the embodied energy and CO_2 emission with respect to respectively the embodied energy and CO_2 burden of a massive vault [%].

	Embodied energy %		CO_2 burden %
Cardboard	88.7	Fusée Céramique	74.5
Fusée Céramique	91.7	Cardboard	78.8
Bamboo	92.6	Bamboo	81.9
Timber logs	128.0	Foam PS	87.3
Foam PS	129.3	Timber logs	97.1
Glass bottles	222.7	Glass bottles	124.3
Steel tubes	288.1	Steel tubes	193.8

TABLE 8.6 Ranking the infill for the embodied energy and CO_2 burden with respect to a massive vault with a depth of 110 mm.

The cardboard tubes and Fusée's Céramique elements perform quite well. Possible the embodied energy and emission of the fusées can be reduced in a hot and dry climate further, in case the fusées are sun dried and not forcibly dried in an oven. Especially for tropical areas the energy needed for production and transport of bamboo is low, for these areas an infill of bamboo can be a good solution.

The embodied energy for timber logs seems very huge; probably the energy needed for production and transport is smaller in wooded areas such as Norway, Finland, Sweden and Germany.

§ 8.7 Prefabrication

Prefabrication of the vault increases the quality and can reduce the production cost. The prefabricated elements are made in a hall with a more or less controlled inner climate so the production is not disturbed by bad weather. Further the construction of a prefabricated vault is much safer for the construction workers. Otherwise the cost and environmental load of the transport of the prefabricated elements can be high if the infrastructure is poor and the factory is far from the site. To speed up the process the liquid concrete is by preference not poured in two stages to prevent uplift. In a production hall it is possible to use a frame to temporarily ballast the infill or to attach the infill firmly to the reinforcement and prevent the uplift till the concrete has set. The moulds made for prefabricated elements can be reused many times, consequently the environmental burden will be much less than for a mould used only once at the building site. An efficient way to save the cost and burden of infill is to make cavities in the concrete by extrusion of the liquid concrete through a nozzle. In the Netherlands the prefabricated hollow core concrete plates are competitive to other systems.
For prefabricated structures the elements are by preference identical and symmetrical with a constant curvature to avoid mistakes in positioning an element not fitting at that position. To produce identical elements the form of the vault is by preference circular. As shown before structurally circular segments are less effective than elements following a parabola. However chapter 1 shows for low rise vaults that the differences between the curvature of a circle segment, parabola and catenary s are small. Furthermore strengthening and stiffening the structure with diagonals will reduce the bending moments and increase the load bearing capacity substantially.
Prefabricated elements can be reinforced, pre- and post-tensioned. Structurally tensioning the reinforcement is very effective. Due to the tensioning, the reinforcement is stressed and the concrete is compressed continuously. Consequently the vault is not cracked or at least less cracked and much stiffer than a reinforced but not tensioned structure. Generally prefabricated hollow core flat elements are pre-tensioned and made in a production hall by extrusion. These elements are nearly flat. For a pre-tensioned curved element the tensioned steel strands, following the curvature of the vault, must be supported well. Due to the curvature the supports of the strands have to resist a substantial downward force. Curved hollow core elements can be post-tensioned too; the strands are positioned in gains and tensioned when the liquid concrete is set and strong enough to resist the tensioning.

§ 8.8 Positioning tubes perpendicular to the span

Generally tubes of cardboard or ceramic are produced without a curvature. To follow the curvature of a vault the elements have to be bent. Bending tubes of cardboard will cause dimpling and cracks. To follow the curvature the tubes must be facetted and jointed with an inclination. Concerning production and assembly, it is much easier for cylindrical vaults to position straight elements perpendicular to the span. Structurally positioning tubes perpendicular to the span can increase the shear stresses and will cause bending stresses. The structure will transfer the loads as a so-called Vierendeel-truss.

A Vierendeel-truss subjected to a lateral load will be subjected to bending moments; these moments will cause compressive and tensile stresses in the upper and lower flange. Due to the punched web these stresses are slightly higher than the stresses in a massive beam. A structure subjected to bending moments is subjected to shear too. To transfer the shear force over the punched web of the Vierendeel truss, the chords and struts will be subjected to bending moments.

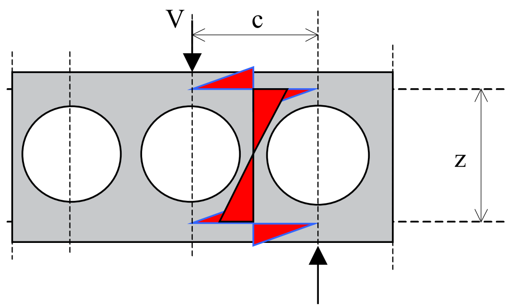

FIGURE 8.8 Transfer of the shear force for a punched structure.

For a structure with tubular infill elements with radius r and centre to centre distance c the bending moments acting on the flanges due to the shear force V_d are approximately equal to:

$$M_{fl} = \tfrac{1}{4} V c$$

The struts between the infill have to resist a bending moment two times the bending moment acting on the flanges:

$$M_{strut} = \tfrac{1}{2} V c \qquad [8.24]$$

Furthermore the strut between the infill has to resist the shear force, $V_{strut} = dM/dx$. Between two tubes the shear force V_{strut} is equal to the horizontal force H. This force follows from the sum of the bending moments acting at the strut divided by the lever arm z:

$$V_{strut} = \Sigma M_{strut} /z \quad \rightarrow \quad V_{strut} = 2 \times (\tfrac{1}{2} V c)/z \quad \rightarrow \quad V_{strut} = V c/z \qquad [8.25]$$

Next the mean shear stress, acting at the strut with a thickness t and width b follows from:

$$\tau_{mean} = V_{strut}/(b\,t) \qquad [8.26]$$

Vierendeel trusses for vaults

Actually the Vierendeel truss is very efficient for form-active vaults. Generally form active vaults are subjected to normal forces. Due to the infill the normal force is transferred by the upper and lower part of the section. The compressive normal stresses at the upper and lower side are larger than the stresses acting on a massive section. The vault will be not cracked if the normal compressive stresses are larger than the tensile bending stresses. The stiffness of an uncracked section is much larger than the stiffness of a cracked section. Increasing the normal compressive stresses can increase the stiffness of the vault, consequently for a concrete vault adding infill can increase the stiffness. For vaults the bending moments are minor, so the shear forces are quite small too, even if the vault is subjected to a concentrated load. This will be shown for the following Vierendeel truss, subjected to two concentrated loads acting at a distance c just beside the centre.

Model

A prefabricated vault is composed of two segments. To simplify construction and assembly the radius of the vault is constant, the line of the system follows a circle segment. Of course for circle segments the load transfer is less effective than for a parabola or a catenary. However for low rise vaults the differences are insignificant. The following chapter describes the load transfer for vaults with a constant radius, strengthened with ties. The circle segment with centre at the top is described with polar coordinates, the x-coordinate and y-coordinate follow from:

$$x = R \sin \phi \text{ and } y = R\,(1 - \cos \phi) \qquad [8.27]$$

The span of the vault is equal to l = 2 a. The coordinates of the supports are calculated with expression [8.28] with $\phi = 2\beta$, so $a = R \sin(2\beta)$.

The rise f follows from: $f = R \times [1 - \cos(2\beta)]$ [8.28]

The angle β follows from: $\tan \beta = f/a$ [8.29]

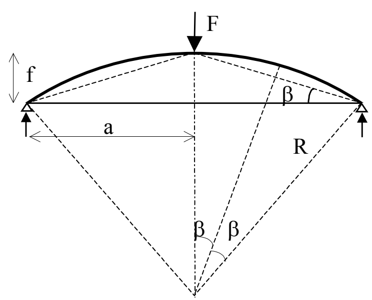

FIGURE 8.9 Prefabricated vault, subjected to a concentrated load.

Assume for this model a width equal to b = 200 mm, a height h = 120 mm, a radius R = 1.905 m and an angle β equal to β = 15° thus 2 β = 30^0. Infill elements Ø60-100, so the centre to centre distance c is 100 mm. The length of the half vault is equal to s = 2 R β. Substituting R = 1.905 m and β = 15^0 gives s = 1.0 m, thus with c = 100 mm the number of infill elements is equal to n = 10. The vault is subjected to a concentrated load F = 10.0 kN acting at the top. The span l follows from l = 2 a = 2 R sin 30^0 = 1.905 m. The rise f follows from [8.28]:

$$f = R\,(1 - \cos 30^0) = 0.255 \text{ m.}$$

For the statically determinate three hinged vault the thrust follows from:

$$H = F\,a/f = \tfrac{1}{2} \times 10.0 \times \tfrac{1}{2} \times 1.905/2/0.255 = 18.67 \text{ kN}$$

Actually the two-hinged vault is statically indeterminate. Chapter 9 describes the load transfer for circular statically indeterminate vaults. The thrust follows from expression [9.38']:

$$H = \frac{\tfrac{1}{2} F\,[\tfrac{1}{2}\sin^2(2\beta) + \cos(2\beta) \times [1 - 2\beta \sin(2\beta)] - \cos^2(2\beta)\,]}{\beta + 2\beta\cos^2(2\beta) - {}^3/_2 \sin(2\beta) \times \cos(2\beta)}$$

Substituting β = 15° and ½ F = 5.0 kN gives:

$$H = \frac{5.0 \times [\,\tfrac{1}{2}\sin^2(30) + \cos(30) \times [1 - (\pi/6) \times \sin(30] - \cos^2(30)]}{(\pi/12) \times (1 + 2\cos^2(30)) - {}^3/_2 \sin(30) \times \cos(30)} = 14.36 \text{ kN}$$

The bending moment follows from expression [9.39];

$$M_{\phi=0} = \underline{\tfrac{1}{2} F R \sin(2\beta) - H R(1-\cos(2\beta))}$$

Substituting $\beta = 15°$ and ½ F = 5.0 kN gives:

$$M_{\phi=0} = \underline{5.0 \times 1.905 \times \sin(30) - 14.36 \times 1.905 \times [1-\cos(30)]} = 1.1 \text{ kN}$$

The shear force acting at the struts follows from (8.25): $V_{strut} = V\, c/z$

Substituting V = ½ × 10.0 kN, z = 0.09 m and c = 0.1 m gives: $V_{strut} = 5.0 \times 0.1/0.09 = 5.556$ kN

The shear stress follows from [8.26]: $\tau_{mean} = V/(b\,t) = 5556/(200\times40) = 0.69$ MPa

Due to the curvature the normal forces and shear forces acting at the chords of the vierendeel truss decrease from the top to the support.

$$N_\phi = N_x \cos\phi + V_x \sin\phi \quad \text{and} \quad V_\phi = -N_x \sin\phi + V_x \cos\phi$$

Computer analysis

For the model an analysis with a computer program, Matrixframe, is made. The thickness of the vertical struts of the Vierendeel truss is equal to 100 − 60 = 40 mm. The lever arm follows from z = 0.06 + 2×0.03/2 = 0.09 m. The radius of the upper and lower chord of the Vierendeel truss are respectively: R_u = 1.905 + 0.09/2 = 1.95 m and R_l = 1.905 − 0.09/2 = 1.86 m. Table 8.7 shows the results.

Node	X =	Y =	Member	nodes	M	N	V	Member.	nodes	M	N	V
1	0	0	1	1-2	0.14	-17.55	2.26	21	1-12	0.14	-2.22	3.12
2	0.102	0,003	2	2-3	0.11	-13.10	1.94	22	2-13	0.21	-0.43	4.55
3	0.204	0.011	3	3-4	0.09	-9.10	1.69	23	3-14	0.19	0.35	4.08
4	0.305	0.024	4	4-5	0.07	-5.81	1.36	24	4-15	0.15	0.11	3.38
5	0.405	0.043	5	5-6	0.06	-3.10	1.09	25	5-16	0.13	-0.12	2.76
6	0.505	0.066	6	6-7	0.05	-1.09	0.77	26	6-17	0.10	-0.16	2.07
7	0.603	0.095	7	7-8	0.03	-0.22	0.43	27	7-18	0.06	-0.31	1.34
8	0.699	0.130	8	8-9	0.02	0.85	0.16	28	8-19	0.03	-0.31	0.65
9	0.793	0.169	9	9-10	0.02	0.85	0.16	29	9-20	0	-0.34	0.01
10	0.885	0.261	10	10-11	0.03	0.27	0.41	30	10-21	0.03	-0.30	0.60
11	0.975	0.09	11	12-13	0.14	6.17	2.41	31	11-22	0.01	0.41	0.28
12	0	0.093	12	13-14	0.12	1.52	2.16					
13	0.097	0.100	13	14-15	0.09	-2.68	1.82					
14	0.194	0.113	14	15-16	0.07	-6.15	1.47	support				
15	0.291	0.131	15	16-17	0.07	-8.97	1.23	1	H =		-20.6	
16	0.387	0.153	16	17-18	0.05	-11.10	0.62	2	H =		9.4	
17	0.481	0.181	17	18-19	0.04	-12.48	0.48	22	H =		11.2	
18	0.575	0.214	18	19-20	0.02	-13.14	0.16	22	V =		5.0	
19	0.667	0.251	19	20-21	0.02	-13.15	0.18					
20	0.757	0.293	20	21-22	0.03	-12.53	0.47					
21	0.844	0.207										
22	0.930	0.228										

TABLE 8.7 Computer analysis of the Vierendeel truss: geometry , coordinates, forces and bending moments

TOC

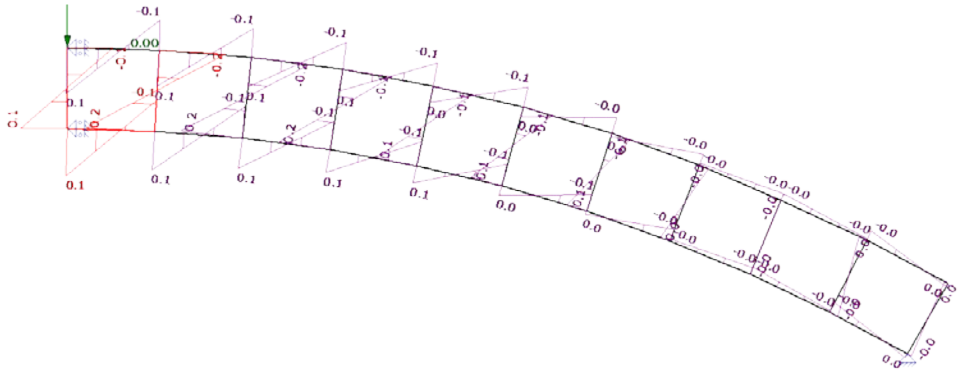

FIGURE 8.10 Half of the Vierendeel truss.

The vertical reaction force and thrust acting at the support node 22 are respectively equal to: V = 5.0 kN and H = 11.2 kN. At the top the horizontal forces are respectively H_1 = 20.6 kN and H_{12} = + 9.4 kN. The bending moment at the top is equal to:

$$M = (H_1 - H_{12}) \times ½\, z \quad \rightarrow \qquad M = (20.6 + 9.4) \times ½ \times 0.090 = 1.35 \text{ kNm}$$

The shear force acting at a flange is at maximum for member 11, this shear force is equal to: V_{fl} = 2.41 kN. According to [8.26] the shear stress is:

$$\tau_{mean} = \frac{V}{b\,t} = \frac{2410}{200 \times 30} = 0.4 \text{ MPa}$$

The shear force acting at a strut is at maximum for member 22, this shear force is equal to: V_{fl} = 4.55 kN. According to [8.26] the shear stress is:

$$\tau_{mean} = \frac{V}{b\,t} = \frac{4550}{200 \times 40} = 0.57 \text{ MPa}$$

The shear forces and stresses approached with the previous simple calculation are slightly larger than the shear forces and stresses calculated with the computer analysis. The approach is on the safe side and can be used to design the vaults in an early stage of the design process.
Comparing the analysis with the computer and the previous analysis of the vault shows that the thrust H = 11.2 kN, according to the computer analysis, is smaller than the thrust calculated previously for the vault H = 14.36 kN. The modelling causes this difference. The height of the Vierendeel truss is 120 mm; the radius of the lower chords is R_{low} = 1.86 m. The span is equal to *l* = 2 a = 2 × 0.93 m this span is smaller than the span of the vault calculated previously with 2 a with a = 0.9525 m. Further at the top the Vierendeel is subjected to two horizontal forces H_1 = 20.6 kN and H_{12} = -9.4 kN. Due to the bending moment M = 1.35 kNm the thrust H = 11.2 kN acts at the top with an eccentricity e = M/H = 0.12 m. So the rise is for the Vierendeel practically equal to:

$$f' = + R_{low} [1 - \cos(2\beta)] + ½ \times z$$

Substituting, R_{llow} = 1.86 m and z = 0.09 m gives for f' and f'/f:

$$f' = 1.86 \times (1 - \cos 30°) + ½ \times 0.09 = 0.294 \text{ m}; \qquad f'/f => 0.294/0.255 = 1.15$$

Due to the increase of the rise and the decrease of the span the thrust calculated with the computer analysis is smaller than the thrust calculated for the vault previously. For a vault with a larger span the effect of the thickness of the Vierendeel will be much smaller.

Table 8.7 and figure 8.10 show that the bending moments due to the concentrated load decrease from the top to the support. This conforms to theory. Due to the curvature, the normal forces and shear forces acting at the chords of the Vierendeel truss decrease from the top to the support with respect to ϕ.

$$N_\phi = N_x \cos\phi + V_x \sin\phi \quad \text{and} \quad V_\phi = -N_x \sin\phi + V_x \cos\phi$$

The system with tubes perpendicular is validated further with experiments. These tests are described in chapter 10.

Conclusions

Due to the curvature of the structure the shear forces and bending moments are decreasing from the loaded node to the supports. The bending stresses and shear stresses in the weakened sections above and adjacent to the tubes are pretty small. The tubes do not affect the load bearing capacity of the vault, if the maximum shear stress between the tubes is smaller than the ultimate stress.
Due to the infill elements the normal compressive stresses will be larger than the normal stresses calculated for a massive section. The vault will be uncracked if the normal compressive stress compensates the tensile bending stress. Thus adding the infill elements increases the normal compressive stresses and increases the stiffness of the vault.

§ 8.9 Selection

The demonstrated possibilities have advantages and disadvantages. The site, the specific design specifications, and the preferences of the designers and the demands of the government affect the selection.
As stated before prefabrication is often lucrative, provided heavy equipment is available, the infrastructure is good, and the building site is not far from the production hall. For a building to be constructed at a remote site, for example on a small island far from any prefab workshop, the vault can be made in situ or with prefabricated elements produced in moulds in a temporary workshop at the side. The choice of the infill elements depends on the availability of the materials too. Transport over long distances will increase the cost, the embodied energy and the CO_2 emission. In some tropical areas bamboo will be available easily, in other areas bundles of cardboard tubes, reeds or branches of willows will be available near the site and can be used as good substitute. Using cassettes to integrate the infill elements with the mould will help to speed up production and reduce the environmental burden. Otherwise the thickness of the vault with cassettes will be larger than for a vault with tubular infill elements. As shown before TT-elements have some disadvantages, the height and specific surface are both less than the height and specific surface of vaults with infill elements, generally the cost of TT-elements are higher than the cost of prefabricated hollow core elements. Structurally elements constructed with infill elements are stiffer and stronger than TT-elements, so these elements can be dimensioned thinner. If the inner space has to be lighted by daylight than it can be interesting to use infill elements with a height equal to the height of the vault. After construction the infill elements are removed and lights are placed in the openings. Structurally this vault with light openings can be considered as a grid of beams.
The selection of the infill elements is also affected by the construction. In the past the Fusée Céramique vaults were made at the building site. To prevent uplift, the liquid concrete was poured in two layers. To follow the curvature the fusées were not very long and jointed in the mould. Nowadays

structures are by preference prefabricated. The moulds of a prefabricated vault can be reused many times and this will reduce the waste. To avoid mistakes and increase the repetition the elements are by preference circularly curved. To reduce the weight and footprint the prefabricated mould can be TT elements, extruded hollow cores, or tubular infill elements of an organic material as bamboo, timber, cardboard or reed. The depth of bamboo branches will vary, this will effect the technique of production and the quality of the products.

To prevent uplift by the liquid concrete the following possibilities arise:

- The concrete is poured in two stages, the infill elements are pushed in the first layer;
- The concrete is poured in one stage, the cavities are made by extrusion;
- The concrete is poured in one stage; the tubular infill elements are connected well to the mould to resist the uplift.
- The concrete is poured in one stage; cassettes are integrated to the mould and reused many times.

Structurally the infill elements must not be very stiff. For the vaults with Fusée Céramique elements the concrete sections were tensioned due to the time dependent deformations. By preference the concrete vault is compressed. To avoid a transfer of the load due to the time dependent deformations the infill has to be flexible.

For cardboard the Young's modulus is quite small, $E \approx 1000\ N/mm^2$, much smaller than the young's modulus of concrete, so the effect of the time dependent deformations is small. The better part of normal load will be transferred by the concrete and reinforcement.
The cardboard tubes decrease the embodied energy and CO_2 emission of the vaults. In the Netherlands cardboard is recycled for 95%. The cost and weight are reasonable. The strength and stiffness are poor, so generally cardboard is not used much for structures, nevertheless the Japanese architect Shigeru Ban designed several projects composed of cardboard tubes structures [McQ08]. In the nineteen fifties cardboard tubes were used as infill elements for floors with a large span to reduce the self-weight. More recently cardboard tubes have been used to the reduce the weight of the floors for the city hall of Almelo [Scha15]. Furthermore the tubes are still used as non-reusable moulds for columns.
Cardboard tubes cannot be curved easily; to follow the curvature of the vault the tubes must be facetted. The tubes do not have to be curved if the tubes are positioned perpendicular to the span. Concerning the production it will be profitable to position the tubes perpendicular to the span. Structurally positioning the tubes perpendicular to the span will increase the normal stresses and the shear stresses. For vaults the increase of the normal stresses will reduce the tensile bending stresses and possibly prevent the structure of cracking and a reduction of the stiffness. Furthermore strengthening the vaults will minimize the bending moments and the shear, consequently for these vaults the structural disadvantage due to the position of the tubes perpendicular to the span, are small.
The following chapter shows the load transfer for vaults, strengthened with ties, that follow a circular segment, with infill elements positioned perpendicular to the span.

9 Prefabricated vaults composed of circle segments

Introduction prefabricated vaults

Prefabrication of a concrete structure will increase the quality and reduce the production cost. Prefabricated elements are made in a workshop with a controlled environment. The fabrication is thus not affected by bad weather. Generally prefabricated structures have to be partitioned into individual elements to allow transportation to the building site. The elements can be quite large if the site is close to a waterway, otherwise the elements have to be small enough to be transported by trucks over roads. By preference prefabricated structures are composed of identical components. This simplifies assembly and reduces the number of moulds required. Due to the constant radius cylindrical vaults can be partitioned easily into identical elements. This reduces the number of required moulds and simplifies the assembly. Structurally circular vaults are best suited to equally distributed radial loads. For any other type of loading the line of the system is not equal to the line of thrust, so the sections are subjected to bending moments. Strengthening the vault with diagonals reduces bending moments and increases the compressive normal forces acting at the sections of the vault. Sufficient strengthening prevents the structure from cracking by bending stresses and also increases the stiffness. This chapter describes the load transfer for prefabricated vaults with a constant radius composed of identical segments. Firstly the transfer of the loads is described for a non-strengthened vault. Next several alternatives are described to strengthen these vaults with diagonals to increase the stiffness and reduce the dimensions, self-weight and the environmental load.

§ 9.1 The description of the coordinates of circle segments

A circular vault with a constant radius can be described using a Cartesian Coordinate system. The following expression describes the vault if the centre of the coordinates is positioned at the crown:

$$x^2 + (y - R)^2 = R^2 \quad \rightarrow \quad y^2 - 2\,R\,y + x^2 = 0 \qquad [9.1]$$

Thus for any x the coordinate y is calculated with: $y = R - (R^2 - x^2)^{1/2}$ [9.2]

The span of the vault is equal to the length of the horizontal chord: $l = 2\,a$. The rise of the vault is equal to f. For a vault with a span $l = 2 \times a$ and a radius R the rise f follows from [9.2]:

$$f = R - (R^2 - a^2)^{0.5}$$

In the same way for a vault with a span $l = 2\,a$ and a rise f the radius R follows from [9.1] :

$$R = \frac{a^2 + f^2}{2\,f} \qquad [9.3]$$

For circular vaults polar coordinates will considerably simplify the analysis. For a circular segment with the centre of the coordinates at the top, the coordinates of a point P at the curve follow from:

$$P(x,y) = (\,R \sin\phi;\ R\,[1 - \cos\phi]\,) \qquad \text{éééé} \qquad [9.4]$$

The coordinates of the supports are found for an angle $\phi = 2\beta$. This angle 2β follows from:

$$\sin(2\beta) = a/R \qquad [9.5]$$

For a given circle segment with radius R the rise f and the span $l = 2a$ follows from:

$$f = R\{1 - \cos(2\beta)\} \quad \rightarrow \quad f = 2R\sin^2\beta \qquad [9.6]$$

$$a = R\sin(2\beta) = 2R\sin\beta\cos\beta \qquad [9.7]$$

The length of half of the arch between the top and a supports with $\phi = 2\beta$ [radians] follows from:

$$s = 2\beta R \qquad [9.8]$$

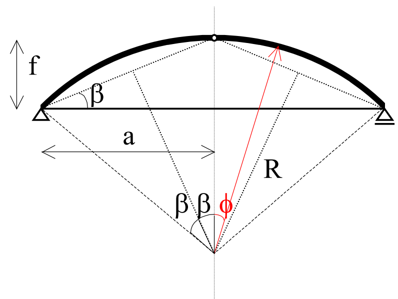

FIGURE 9.1 Segment of a circle described with Polar Coordinate system.

§ 9.2 Three hinged vault following circular segment

A vault is supported by two simple supports and hinged at the top. The structure is statically determined, the reactions, forces and bending moments can be defined using the expressions that show the equilibrium of forces and moments. To simplify the calculations the structure is described with polar coordinates. The angle between the vertical axis and the vector pointing to the support is equal to 2β.

Three hinged vault subjected to an equally distributed load q:

For the three hinged vault subjected to a surface load q the vertical reaction forces acting at the supports ,V_A and V_B, are equal to:

$$V_A = V_B = qR\sin(2\beta) \qquad [9.9]$$

The moment M_0 at the centre due to the distributed load is equal to zero:

$$M_{\phi=0} = VR\sin(2\beta) - Hf - \tfrac{1}{2}qR\sin^2(2\beta) = 0$$

The thrust H is defined by substituting the reaction force V, the rise f and the span a into this expression:

$$H = \frac{\tfrac{1}{2}qR^2(2\sin\beta\cos\beta)^2}{2R\sin^2\beta} \quad \rightarrow \quad H = qR\cos(2\beta) \qquad [9.10]$$

The bending moment M_ϕ for an angle ϕ follows from: $M_\phi = H\,R\,[1 - \cos\phi] - ½\,q\,R^2 \sin^2\phi$

Substituting the thrust H [9.10] into this expression gives:

$$M_\phi = q\,R^2\,[\cos^2\beta\;(1 - \cos\phi) - ½\sin^2\phi] \qquad [9.11]$$

For $\phi = \beta$: $M_{\phi=\beta} = -½\,q\,R^2\,(1 + 2\cos^3\beta - 3\cos^2\beta)$
[9.11']

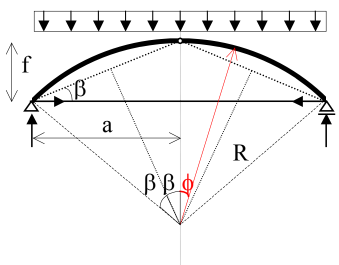

FIGURE 9.2 Vaults subjected to an equally distributed load q.

The maximum moment acting at the vault is found for $dM_\phi/d\phi = 0$, differentiating the expression for the bending moment [9.11] with respect to ϕ gives:

$$dM_\phi/d\phi = q\,R^2\cos^2\beta\,\sin\phi - q\,R^2\sin\phi\,\cos\phi = 0 \qquad \rightarrow \cos\phi = \cos^2\beta$$

The maximum moment is found by substituting $\cos\phi = \cos^2\beta$ and $\sin^2\phi = (1 - \cos^4\beta)$ into the expression [9.11]:

$$M_{\phi\,max} = q\,R^2\,[\cos^2\beta\,(1 - \cos^2\beta) - ½\,(1 - \cos^4\beta)] \qquad \rightarrow M_{\phi\,max} = -½\,q\,R^2\sin^4\beta \qquad [9.12]$$

Three-hinged vault subjected to an equally distributed surface load q

Due to the dead load the arch will be subjected to an equally distributed surface load q. The three-hinged arch is statically determinate. For this loading it is easier to describe the expressions with polar coordinates. The angle between the vertical axis and the radius pointing to the support is equal to 2 β.

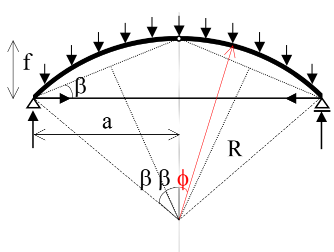

FIGURE 9.3 Segment of a vault subjected to an equally distributed surface load.

For the three hinged arch subjected to a surface load q the vertical reaction force acting at the supports V_A and V_B is equal to:

$$V_A = V_B = \int_{\phi=0}^{\phi=2\beta} q\,R\,d\phi = 2\,\beta\,q\,R \qquad [9.13]$$

The bending moment M_0 acting at the centre is equal to zero:

$$M_{\phi=0} = V R \sin(2\beta) - H f - q R^2 \int_{\phi=0}^{\phi=2\beta} \sin\phi \, d\phi = 0$$

To define the thrust H the reaction force $V = 2\beta q R$ is substituted and the expression is integrated between the constraints $\phi = 0$ and $\phi = 2\beta$:

$$M_{\phi=0} = 2\beta q R^2 \sin(2\beta) - H f - q R^2 [1 - \cos(2\beta)] = 0$$

Now the thrust H follows from: $H = \dfrac{q R [2\beta \sin(2\beta) + \cos(2\beta) - 1]}{(1 - \cos(2\beta))}$

Substituting $\cos(2\beta) = 1 - 2\sin^2\beta$ and $\sin(2\beta) = 2\cos\beta \sin\beta$:

$$H = \frac{q R [2\beta \cos\beta - \sin\beta]}{\sin\beta} \quad [9.14]$$

The bending moment for a certain angle ϕ_1 follows from:

$$M_{\phi 1} = H R [1 - \cos\phi_1] - q R^2 \int_{\phi=0}^{\phi 1} (\sin\phi_1 - \sin\phi) \, d\phi$$

Integrating of this expression between the constraints $\phi = 0$ and $\phi = \phi_1$ gives:

$$M_{\phi 1} = H R [1 - \cos\phi_1] - q R^2 [\phi_1 \sin\phi_1 - \cos\phi_1 + 1]$$

For $\phi_1 = \phi$ the bending moment is:

$$\frac{M_\phi}{q R^2} = \frac{(2\beta \cos\beta - \sin\beta)}{\sin\beta} \times (1 - \cos\phi) - \phi \sin\phi + 1 - \cos\phi \quad \rightarrow$$

$$\frac{M_\phi}{q R^2} = \frac{2\beta \cos\beta}{\sin\beta} \times [1 - \cos\phi] - \phi \sin\phi \quad [9.15]$$

For $\phi = \beta$:

$$M_{\phi=\beta} = -\frac{q R^2 \beta [1 + \cos^2\beta - 2\cos\beta]}{\sin\beta} \quad [9.15']$$

The maximum moment acting at the vault is found for $dM/d\phi = 0$. Differentiating expression [9.15] with respect to ϕ gives for $\phi = \phi_u$:

$$dM_\phi/d\phi = q R^2 \left[\frac{2\beta \sin\phi_u}{\tan\beta} - \sin\phi_u - \phi_u \cos\phi_u \right] = 0 \quad \rightarrow$$

$$\frac{2\beta}{\tan\beta} - 1 = \frac{\phi_u}{\tan\phi_u} \quad [9.16]$$

Three hinged vault subjected to an asymmetric load

Due to an equally distributed asymmetric load the vault is subjected to bending. The assumption is made that the live load q_e is equally distributed and acting on the right side of the vault. The vertical reaction forces acting on the supports are respectively:

$$V_A = \tfrac{1}{4} q a = \tfrac{1}{4} q R \sin(2\beta) \text{ and } V_B = \tfrac{3}{4} q a = \tfrac{3}{4} q R \sin(2\beta)$$

The thrust is calculated for the not loaded half of the vault with the equilibrium of the bending moment around the top.

$$M_{\phi=0} = H f - \tfrac{1}{4} q a^2 = 0 \quad \rightarrow \quad H = \tfrac{1}{4} q a^2/f$$

In the same way the thrust is calculated for the loaded part:

$$M_{\phi=0} = \tfrac{3}{4} q a^2 - H f - \tfrac{1}{2} q a^2 = 0 \quad \rightarrow \quad H = \tfrac{1}{4} q a^2/f$$

Substituting the rise f and the span into the expression for the thrust:

$$H = \frac{\frac{1}{4} q R^2 \sin^2(2\beta)}{R\,[1 - \cos(2\beta)]}$$

Substituting $\cos(2\beta) = 1 - 2\sin^2\beta$ and $\sin(2\beta) = 2\cos\beta\sin\beta$: $H = \frac{1}{2} q R \cos^2\beta$ [9.17]

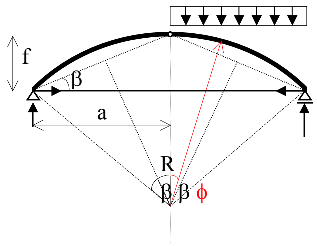

FIGURE 9.4 Vault subjected to an asymmetrical equally distributed load.

For the unloaded side the bending moment M_ϕ is calculated for an angle ϕ from the top with:

$$M_\phi = H R (1 - \cos\phi) - \tfrac{1}{4} q R \sin(2\beta) \times R \sin\phi$$

Substituting $H = \frac{1}{2} q R \cos^2\beta$ and $\sin(2\beta) = 2\cos\beta\sin\beta$:

$$M_\phi = \tfrac{1}{2} q R^2 [\cos^2\beta (1 - \cos\phi) - \sin\beta\cos\beta\sin\phi]$$

[9.18]

The bending moment is maximum for $dM_\phi/d\phi = 0$. Derivate the expression for M_ϕ gives for $\phi = \phi_u$:

$$dM_\phi/d\phi = \tfrac{1}{2} q R^2 [\cos^2\beta \sin\phi_u - \sin\beta\cos\beta\cos\phi_u] = 0 \rightarrow \tan\phi = \tan\beta \rightarrow \phi_u = \beta$$

The maximum bending moment is calculated by substituting $\phi_u = \beta$ in the expression [9.18]:

$$M_{max} = M_{\phi=\beta} = \tfrac{1}{2} q R^2 [\cos^2\beta (1 - \cos\beta) - \sin^2\beta\cos\beta] \rightarrow$$

$$M_{max} = M_{\phi=\beta} = -\tfrac{1}{2} q R^2 \cos\beta (1 - \cos\beta)$$ [9.19]

For the loaded part of the vault the bending moment M_ϕ is calculated for a certain angle ϕ_1 from the top with:

$$M_{\phi 1} = H R (1 - \cos\phi_1) + \tfrac{1}{4} q R \sin(2\beta) R \sin\phi_1 - \tfrac{1}{2} q (R \sin\phi_1)^2$$

Substituting $H = \frac{1}{2} q R \cos^2\beta$ and $\sin(2\beta) = 2\cos\beta\sin\beta$:

$$M_{\phi 1} = \tfrac{1}{2} q R^2 [\cos^2\beta (1 - \cos\phi_1) + \sin\beta\cos\beta\sin\phi_1 - \sin^2\phi_1]$$

For $\phi = \phi_1$ this expression becomes:

$$M_\phi = \tfrac{1}{2} q R^2 [\cos^2\beta (1 - \cos\phi) + \sin\beta\cos\beta\sin\phi - \sin^2\phi]$$

[9.20]

The bending moment is maximum for $dM_f/d\phi = 0$. To define the angle ϕ_u expression [9.20] is derived for $M_f = 0$:

$$dM_\phi/d\phi = \tfrac{1}{2} q R^2 [\cos^2\beta \sin\phi_u + \sin\beta\cos\beta\cos\phi_u - 2\sin\phi_u\cos\phi_u] = 0 \rightarrow$$

$$2\sin\phi_u\cos\phi_u - \cos^2\beta\sin\phi_u = \sin\beta\cos\beta\cos\phi_u$$

Dividing this expression by $\cos\phi_u$ gives: $\cos^2\beta\tan\phi_u = 2\sin\phi_u - \sin\beta\cos\beta$

This expression can be solved numerically. If β is small, then the angle ϕ_u will approach the angle β. Substitution of $\phi_u = \beta$ into expression [9.20] gives for the bending moment M_ϕ the following expression:

$$M_{\phi=\beta} = ½\, q\, R^2 \cos(2\beta) \times (1 - \cos\beta) \qquad [9.21]$$

This moment is positive and slightly less than the bending moment acting at the not loaded part of the vault, see expression [9.19].

Three hinged vault subjected to an anti-metrical load

Due to an anti-metrical load the three hinged vault will be subjected to bending. Again the assumption is made that the arch is simple supported and hinged at the top. The structure is statically determinate. The vertical reaction acting at the support at the right V_B follows from the equilibrium of bending moments for the support at the left:

$$V_B = \frac{q\, a\, (^3/_2\, a - {}^1/_2\, a)}{2\, a} = ½\, q\, a \uparrow$$

The vertical force at the support at the left is equal to: $V_A = -½\, q\, a \downarrow$

The thrust H acting at the supports follows from the equilibrium of the moments around the hinge at the top.

$$H = \frac{½\, q\, a^2 - ½\, q\, a^2}{f} = 0 \qquad [9.22]$$

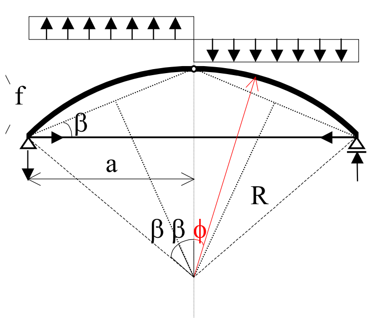

FIGURE 9.5 Vault subjected to an antimetrical load

The bending moment M_x at a distance x from the top follows from: $M_x = ½\, q\, a\, x - ½\, q\, x^2$

Substituting $x = R \sin\phi$ and $a = 2 R \sin\beta \cos\beta$ into the expression for M_x to define M_f:

$$M_\phi = ½\, q\, R^2 (2 \sin\beta \cos\beta \sin\phi - \sin^2\phi) \qquad [9.23]$$

For $\phi = \beta$: $M_{\phi=\beta} = ½\, q\, R^2 \sin^2\beta\, (2\cos\beta - 1)$
[9.23']

The maximum bending moment is found for $dM_\phi/d\phi = 0$, deriving the expression for the bending moment gives:

$$dM_\phi/d\phi = ½\, q\, R^2 (2 \sin\beta \cos\beta \cos\phi - 2 \sin\phi \cos\phi)$$

The maximum bending moment is found when $dM_f/d\phi = 0$:

$$2 \sin\beta \cos\beta \cos\phi_u - 2 \sin\phi_u \cos\phi_u = 0 \quad \rightarrow \quad \sin\phi_u = \sin\beta \cos\beta$$

Substituting ϕ_u into the expression [9.23] gives for the maximum moment:

$$M_{max} = ½\, q\, R^2 \sin^2\beta \cos^2\beta \qquad [9.24]$$

Three hinged vault subjected to a concentrated load acting at the centre

The three hinged vault will be subjected to bending if the arch is loaded by a concentrated load acting at the centre. The assumption is made that the arch is simple supported and hinged at the top. The structure is statically determinate. The vertical reaction acting at the support at the left V_A, and right side V_B, is equal to: $V_A = V_B = ½ F$.
The thrust H acting at the supports follows from the equilibrium of the moments around the hinge at the top.

$$H f - V a = 0 \quad \rightarrow \quad H = ½ F a / f$$

Substituting $a = R \sin(2\beta)$ and $f = R(1 - \cos 2\beta)$:

$$H = \frac{½ F \sin(2\beta)}{1 - \cos(2\beta)}$$

Substituting $\sin(2\beta) = 2 \sin\beta \cos\beta$ and $\cos(2\beta) = 1 - 2\sin^2\beta$ into the expression:

$$H = \frac{F \sin\beta \cos\beta}{2 \sin^2\beta} \quad \rightarrow \quad H = \frac{½ F}{\tan\beta} \qquad [9.25]$$

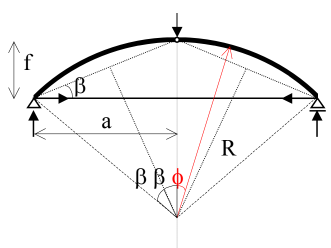

FIGURE 9.6 Vault subjected to a concentrated load.

The bending moment M_x at a distance x follows from: $M_x = H y - ½ F x$

Substituting H, $x = R \sin\phi$ and $y = R(1 - \cos\phi)$ into the expression for M_x to define M_ϕ:

$$M_\phi = \frac{½ F R (1 - \cos\phi)}{\tan\beta} - ½ F R \sin\phi \qquad [9.26]$$

The bending moment is maximum when $dM_\phi/d\phi = 0$, deriving the expression for the bending moment for $\phi = \phi_u$:

$$dM_\phi/d\phi = \frac{½ F R \sin\phi_u}{\tan\beta} - ½ F R \cos\phi_u = 0 \quad \rightarrow \quad \tan\phi_u = \tan\beta \qquad \text{thus: } \phi_u = \beta$$

For this load the bending moment is maximum when $\phi_u = \beta$. The maximum moment follows by substituting $\phi_u = \beta$ into expression [9.22]:

$$M_{\phi=\beta} = \frac{\tfrac{1}{2} F R (1 - \cos\beta)}{\tan\beta} - \tfrac{1}{2} F R \sin\beta \quad \rightarrow \quad M_{\phi=\beta} = -\frac{\tfrac{1}{2} F R (1 - \cos\beta)}{\sin\beta} \qquad [9.27]$$

The bending moment is negative, due to this moment the arch is tensioned at the outer side.

Three hinged vault subjected to a concentrated horizontal force acting at the top

The three hinged vault will be subjected to bending if the arch is loaded by a horizontal force H acting at the top. The assumption is made that the arch is simple supported and hinged at the top. The structure is statically determinate. The horizontal reaction acting at the support at the left H_A, and right side H_B, is equal to: $H_A = H_B = ½ H$. The vertical reaction force V_B acting at the support at the right follows from the equilibrium of the moments around the hinge at the top.

$$½ H f - V_B a = 0 \quad \rightarrow \quad V_B = ½ H f/a \uparrow$$

Substituting $a = R \sin(2\beta)$ and $f = 2 R \sin^2\beta$: $\quad V_B = ½ H \tan\beta \uparrow$

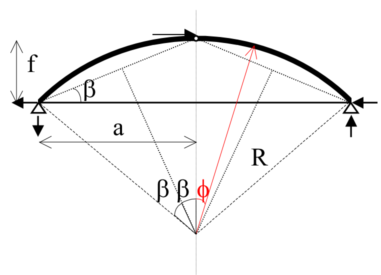

FIGURE 9.7 Vault subjected to a concentrated load.

The bending moment M_x at a distance x follows from: $\quad M_x = ½ H y – V_A x$

Substituting $V_A x = R \sin\beta$ and $y = R (1 - \cos\phi)$ into the expression for M_x to define M_ϕ:

$$M_\phi = ½ H R (1 - \cos\phi - \tan\beta \sin\phi) \qquad [9.28]$$

The bending moment is maximum when $dM_\phi/d\phi = 0$, deriving the expression for the bending moment:

$$dM_\phi/d\phi = ½ H R (\sin\phi_u - \tan\beta \cos\phi_u) = 0 \rightarrow \tan\phi_u = \tan\beta \rightarrow \text{thus: } \phi_u = \beta$$

For this load the maximum bending moment is found when $\phi_u = \beta$. Substituting $\phi_u = \beta$ into expression [9.28] gives:

$$M_{max} = - ½ H R (\cos\beta + \tan\beta \sin\beta – 1)$$

[9.29]

Load	Bending moment, $\phi = \beta$	Maximum bending moment
Equally distributed load:	$M_{\phi=\beta} = -\frac{1}{2} q R^2 [1 + 2\cos^3\beta - 3\cos^2\beta]$	$M_{max} = -\frac{1}{2} q R^2 \sin^4\beta$
Equally distributed surface load	$M_{f\phi=\beta} = \frac{-q R^2 \beta (1 + \cos^2\beta - 2\cos\beta)}{\sin\beta}$	
Asymmetrical load	$M_{\phi=\beta} = \frac{1}{2} q R^2 \cos(2\beta) \times (1 - \cos\beta)$	
Anti-metrical load:	$M_{\phi=\beta} = \frac{1}{2} q R^2 \sin^2\beta \times [2\cos\beta - 1]$	$M_{max} = \frac{1}{2} q R^2 \sin^2\beta \cos^2\beta$
Concentrated vertical load:	$M_{\phi=\beta} = \frac{-\frac{1}{2} F R (1 - \cos\beta)}{\sin\beta}$	$M_{max} \frac{-\frac{1}{2} F R (1 - \cos\beta)}{\sin\beta}$
Concentrated horizontal load:	$M_{\phi=\beta} = -\frac{1}{2} H R (\cos\beta + \tan\beta \sin\beta - 1)$	$M_{max} = -\frac{1}{2} H R (\cos\beta + \tan\beta \sin\beta - 1)$

TABLE 9.1 Bending moments for a three hinged vault following a circle segment

Table 9.1 shows for varying loads the bending moments for a three hinged vault following a circle segment.

§ 9.3 T wo hinged vaults

Two hinged vaults are statically indeterminate. To define the thrust the structure is transferred into a statically determinate vault by changing one of the supports into a roller and loading this support with a horizontal force H. The displacement of this support due to the force H and the vertical load is defined for the load and a horizontal force H. For a statically indeterminate vault the horizontal displacement of the roller support must be zero. Consequently the displacement of the load has to be equal to the displacement of the horizontal force. This equation gives the horizontal force H for the statically indeterminate vault. Generally the force H is smaller than the thrust defined for the statically determinate vault. The displacement of the roller support follows from:

$$\Delta = \int \frac{M (f - y)\, ds}{EI} \quad [9.30]$$

Due to the force H acting at the roller the bending moment, for the curved element with centre at the top, follows from:

$$M_x = H (f - y)$$

Substituting this bending moment into expression [9.30] defines the displacement:

$$\Delta = \frac{H}{EI} \int_{\phi=0}^{\phi} (f - y)^2\, ds$$

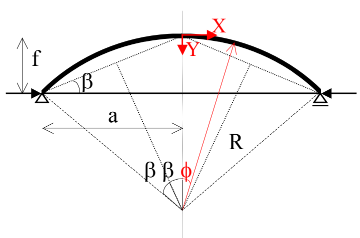

FIGURE 9.8 Statically indeterminate curved element subjected to a horizontal force H acting at the roller support.

For circle segments it is more convenient to use polar coordinates. For a segment with the centre at the top the coordinates are $x = R \sin \phi$ and $y = R (1 - \cos \phi)$. Further the span and rise are equal to respectively $a = R \sin (2\beta)$ and $f = R [1 - \cos (2\beta)]$. The length of an infinitive small part of the vault is equal to: $ds = R\, d\phi$. Substituting the polar coordinates into the expression for the displacement Δ gives:

$$\Delta_H = \frac{H R^3}{EI} \int_0^{\phi=2\beta} [\cos\phi - \cos(2\beta)]^2 d\phi$$

Integrating this expression between $\phi = 0$ and the angle $\phi = 2\beta$ gives:

$$\Delta_H = \frac{H R^3}{EI} [\beta + 2\beta \cos^2(2\beta) - {}^3/_2 \sin(2\beta)\cos(2\beta)] \qquad [9.31]$$

The displacement of the supports is equal to the lengthening of the tie between the supports. Generally this deformation is much smaller than the deformation of the vault. The lengthening of the tie is equal to:

$$\Delta_T = \frac{H R \sin(2\beta)}{EA_T} \qquad [9.32]$$

Next the deformation Δ is defined for the load. The thrust H follows from the equation:

$$\Delta - \Delta_H = \Delta_T. \qquad [9.33]$$

Successively the thrust H is defined for an equally distributed load q and a concentrated load F acting at the top.

Two hinged vault subjected to an equally distributed load q:

The assumption is made that the vault is supported by a hinge and a roller. For a curved element subjected to an eqully distributed load q the bending moment follows from:

$$M_x = \tfrac{1}{2} q (a^2 - x^2)$$

Substituting polar coordinates gives: $M_x = \tfrac{1}{2} q R^2 [\sin^2(2\beta) - \sin^2\phi]$

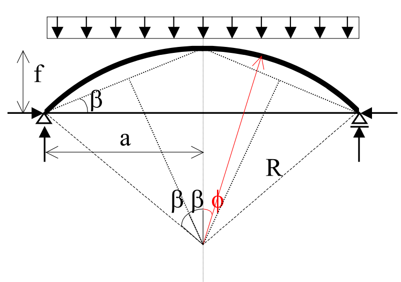

FIGURE 9.9 Figure 9.9: Vaults subjected to an equally distributed load q.

The displacement of the roller support follows from [9.30]:

$$\Delta_q = \int_0^{\phi=2\beta} \frac{M_x (f-y)\, ds}{2\, EI}$$

Substituting M_x, f, y and s gives:

$$\Delta_q = \frac{q R^4}{2 EI} \int_0^{\phi=2\beta} [\sin^2(2\beta) - \sin^2\phi] \times [\cos\phi - \cos(2\beta)]\, d\phi$$

Integrating this expression for the deformation between $\phi = 0$ and $\phi = 2\beta$ gives:

$$\Delta_q = \frac{q R^4}{2 EI} \{{}^7/_6 \sin^3(2\beta) + \beta \cos(2\beta) \times [1 - 2\sin^2(2\beta)] - ½ \sin(2\beta)\} \quad [9.34]$$

Next the thrust is defined with [9.31], [9.32] , [9.33] and [9.34]:

$$H = ½ q R \frac{\{{}^7/_6 \sin^3(2\beta) + \beta \cos(2\beta) \times [1 - 2\sin^2(2\beta)] - ½ \sin(2\beta)\}}{\sin(2\beta) EI/(R^2 EA_T) + \beta + 2\beta \cos^2(2\beta) - {}^3/_2 \sin(2\beta)\cos(2\beta)} \quad [9.35]$$

Generally the deformation of the tie is much smaller than the deformation of the vault. Neglecting the deformation of the tie gives:

$$H = ½ q R \frac{\{{}^7/_6 \sin^3(2\beta) + \beta \cos(2\beta) \times [1 - 2\sin^2(2\beta)] - ½ \sin(2\beta)\}}{\beta + 2\beta \cos^2(2\beta) - {}^3/_2 \sin(2\beta) \cos(2\beta)} \quad [9.35']$$

The vertical reaction forces acting at the supports V_A and V_B are equal to: $V_A = V_B = q R \sin(2\beta)$

The bending moment $M_{f=0}$ at the centre due to the distributed load q follows from:

$$M_{\phi=0} = ½ q R^2 \sin^2(2\beta) - H R [1 - \cos(2\beta)]$$

[9.36]

Two hinged vault subjected to a concentrated vertical force F acting at the top:

The assumption is made that the vault is supported by a hinge and a roller. For a curved element subjected to a concentrated load acting at the top the bending moment follows from: $M_x = ½ F (a - x)$

Substituting polar coordinates gives: $M_x = ½ F R [\sin(2\beta) - \sin\phi]$

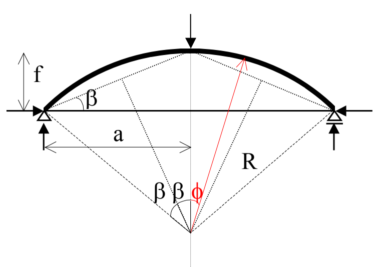

FIGURE 9.10 Vault subjected to a concentrated force F.

The deformation of the roller support follows from:

$$\Delta_F = \frac{F R^3}{2 EI} \int_0^{\phi=2\beta} \{\sin(2\beta) - \sin\phi] \times [\cos\phi - \cos(2\beta)\}\, d\phi$$

Integrating this expression for the deformation between $\phi = 0$ and $\phi = \phi_1$ gives:

$$\Delta_F = \frac{F R^3}{2 EI} \{ ½ \sin^2(2\beta) + \cos(2\beta) \times [1 - 2\beta \sin(2\beta)] - \cos^2(2\beta)\} \quad [9.37]$$

Next the thrust is defined with [9.31], [9.32], [9.33] and [9.37]:

$$H = \frac{\frac{1}{2} F \times \{ \frac{1}{2} \sin^2(2\beta) + \cos(2\beta)\times[1- 2\beta \sin(2\beta)] - \cos^2(2\beta) \}}{\sin(2\beta) EI/(R^2 EA_T) + \beta + 2\beta \cos^2(2\beta) - {}^3/_2 \sin(2\beta)\cos(2\beta)} \qquad [9.38]$$

Generally the deformation of the tie is much smaller than the deformation of the vault. Neglecting the deformation of the tie gives:

$$H = \frac{\frac{1}{2} F\times \{ \frac{1}{2} \sin^2(2\beta) + \cos(2\beta)\times[1- 2\beta \sin(2\beta)] - \cos^2(2\beta) \}}{\beta + 2\beta \cos^2(2\beta) - {}^3/_2 \sin(2\beta)\cos(2\beta)} \qquad [9.38']$$

The vertical reaction forces acting at the supports V_A and V_B are equal to: $V_A = V_B = \frac{1}{2} F$

The bending moment $M_{\phi=0}$ at the centre follows from:

$$M_{\phi=0} = \tfrac{1}{2} F R \sin(2\beta) - H R [1- \cos(2\beta)] \qquad [9.39]$$

§ 9.4 Example prefabricated three hinged vault composed of circular segments

A prefabricated vault is composed of segments, following a part of a circle. The span is equal to l = 14.4 m, the rise is of the swallow vault is equal to f = l/8 = 1.8 m. The variable load acting on the vault is equal to 5.0 kN/m². The elements of the vault are described with polar coordinates. For a vault with a span l = 2 a and a rise of the radius R follows from [9.3]:

$$R = \frac{a^2 + f^2}{2f} = \frac{7.2^2 + 1.8^2}{2 \times 1.8} = 15.3 \text{ m}$$

The coordinates of the supports are found for an angle $\phi = 2\beta$. The angle β follows from [9.5]:

$\sin 2\beta = a/R = 7.2/15.3$ $\quad 2\beta = 28.072^0$ or $2\beta = 28.0725^0 \times \pi/180 = 0.49$ rad

The vault is constructed with a rectangular section with a height of 110 mm. To reduce the weight and environmental load, cardboard tubes Ø60 mm are positioned with a centre to centre distance of 90 mm perpendicular to the span.For the section with a width of 1.0 m the volume, area and second moment of the area are:

Volume:	V_c =	$110\times1000 - 11.1 \times\pi \times 30^2$ =	78.6×10^3	mm²
Area:	A_c =	$(110 - 60) \times 1000$ =	50×10^3	mm²
Second moment of the Area;	I_c =	$1000\times110^3/12 - 1000\times60^3/12$ =	92.917×10^6	mm⁴

Loads	vault:	0.0786 × 24 =	1.9 kN/m²
	finishing:		0.3 kN/m²
	soil, vegetation:		1.0 kN/m²
	dead load:		3.2 kN/m²
	live load:		5.0 kN/m²

Permanent load

Due to the dead load the vault with a width of 1.0 m is subjected to a surface load equal to q_g = 3.2 kN/m.The vertical reaction acting at the supports V_A and V_B follows from (9.12):

$$V_A = V_B = 2\,\beta\, q\, R = 0.49 \times 3.2 \times 15.3 = 24 \text{ kN}$$

The thrust H follows from [9.14]:

$$H = \frac{q_g R\,[2\,\beta \cos\beta - \sin\beta]}{\sin\beta} = \frac{3.2 \times 15.3 \times [0.49 \times \cos(14.036) - \sin(14.036)\,]}{\sin(14.036)} = 47 \text{ kN}$$

The vault is subjected to the maximum moment for an angle ϕ, this angle follows from [9.16]:

$$\frac{2\,\beta}{\tan\beta} - 1 = \frac{\phi}{\tan\phi} \quad \rightarrow \quad \frac{0.49}{\tan(14.036)} - 1 = \frac{\phi}{\tan\phi} \quad \rightarrow \quad \phi = 20^0$$

The bending moment for $\phi = 20^o$ and $\beta = 14.036^o$ follows from [9.15]:

$$\frac{M_\phi}{q\,R^2} = \frac{2\,\beta}{\tan\beta}\,[1 - \cos\phi] - \phi \sin\phi \quad \rightarrow \quad M_\phi = -\,0.001195 \times 3.2 \times 15.3^2 = -0.9 \text{ kNm}$$

Live load

For the three hinged arch subjected to a live load q_e = 5.0 kN/m the vertical reaction acting at the supports V_A and V_B follows from [9.9]:

$$V_A = V_B = q_e R \sin(2\,\beta) = 5.0 \times 15.3 \times \sin(28.072) = 36.0 \text{ kN}$$

The thrust follows from [9.10]

$$H = q_e R \cos^2\beta = 5.0 \times 15.3 \times \cos^2(14.036) = 72.0 \text{ kN}$$

The maximum bending moment is found for $\cos\phi = \cos^2\beta$, substituting $\phi = 19.75^0$ into [9.12] gives:

$$M_{\phi\ max} = -\,½\, q_e\, R^2 \sin^4\beta = 2.024 \text{ kNm}$$

Asymmetric live load

The vertical and horizontal reaction force acting on the supports are respectively:

$$V_A = ¼\, q_e\, a = ¼\, q_e\, R \sin(2\,\beta) = 9.0 \text{ kN}$$

$$V_B = ¾\, q_e\, a = ¾\, q_e\, R \sin(2\,\beta) = 27.0 \text{ kN}$$

The thrust follows from [9.17]:

$$H = ½\, q_e R \cos^2\beta = ½ \times 5.0 \times 15.3 \times \cos^2(14.036) = 36.0 \text{ kN}$$

For the unloaded side the maximum bending moment M_f is found for $\phi = \beta$ and follows from [9.19] :

$$M_{\phi=\beta} = -\,½\, q\, R^2 \cos\beta\,(1 - \cos\beta) = 16.95 \text{ kNm}$$

For the loaded part of the vault the bending moment M_f is maximum for

$$\cos^2\beta \tan\phi_u = 2 \sin\phi_u - \sin\beta \cos\beta \quad \rightarrow \quad \phi = 13^o$$

The bending moment is calculated with [9.20], substituting $\phi = 13^0$ into [9.20] gives:

$$M_\phi = ½\, q\, R^2 [\,\cos^2\beta\,(1 - \cos\phi) + \sin\beta \cos\beta \sin\phi - \sin^2\phi] = 15.5 \text{ kNm}$$

Load	H	F	V	N	σ = N/A	M	σ = M/W
Perm. load	47.0 kN	20⁰	17.1 kN	50.0 kN	-0.95 MPa	0.90 kNm	0.53 MPa
Asym.live load	36.0 kN	14.04	9.0 kN	37.1 kN	-0.74 MPA	16.95 KNm	10.04 MPa
Asym.live load	36.0 kN	14.04	9.0 kN	37.1 kN	-0.74 MPa	15.50 KNm	9.18 MPa

TABLE 9.2 The stresses, due to the permanent and asymmetric load acting at the three hinged vault:

Analysis with computerprogram

To validate the elementary analysis the three hinged vault, subjected to the permanent load q_g= 3.2 kN/m and the asymmetrical live load q_e = 5.0 kN/m, is analysed with a FEM computer program (Matrixframe).

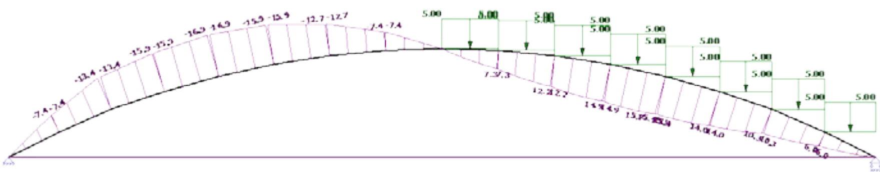

FIGURE 9.11 Bending moments due to the variable load

node	x-coord.	y-coord.	member	dead load N	dead load M	asym live load N	asym live load M	asym live load V
n1	-7.20	0.0	M1:n1-n2	-54.8	0.61	-36.3	0.74	7.9
n2	-6.360	-0.415	M2: n2-n3	-51.5	1.85	-36.6	13.44	6.4
n3	-5.497	-0.779	M3; n3-n4	-50.3	1.85	-37.0	15.88	2.6
n4	-4.612	-1.088	M4: n4-n5	-49.3	0.67	-37.1	16.95	1.1
n5	-3.711	-1.343	M5: n5-n6	-48.5	0.67	-37.1	16.95	1.2
n6	-2.795	-1.542	M6: n6-n7	-47.8	0.41	-37.0	15.87	3.4
n7	-1.869	-1.685	M7: n7-n8	-47.4	0.24	-36.7	12.68	5.7
n8	-0.937	-1.771	M8:n8-n9	-47.1	0.33	-36.3	7.39	7.9
n9	0	-1.80	M9:n9-n10	-47.1	0.33	-35.9	7.28	10.1
n10	0.937	-1.771	M10: n10-n11	-47.4	0.24	-35.9	12.23	7.6
n11	1.869	-1.685	M11: n11-n12	-47.8	0.41	-36.3	14.91	5.2
n12	2.795	-1.542	M12: n12-n13	-48.5	0.67	-37.2	15.42	2.8
n13	3.711	-1.343	M13: n13-n14	-49.3	0.67	-38.5	15.42	3.7
n14	4.612	-1.088	M14: n14-n15	-50.3	1.52	-40.0	13.96	6.1
n15	5.479	-0.799	M15: n15-n16	-51.5	1.52	-42.1	10.30	6.5
n16	6.360	-0.415	M16: n16-n17	-52.8	0.61	-44.2	5.98	8.3
n17	7.20	0	M17:n17-n18	+47.0		+36.0		
n18	0	0	M18:n1-n18	+47.0		+36.0		+1.3

TABLE 9.3 Coordinates, forces and bending moments for the vault subjected to a surface load p_g = 3.2 kN/m and an asymmetrical load q_e = 5.0 kN/m

Load	V =	N =	σ = N/A	M =	σ = M×½ t/I
Permanent load		51.5 kN	1.03 MPa	-1.85 kNm	1.10 MPa
asym. live load	9.0 kN	37.1 kN	0.74 MPa	16.95 kNm	10.03 MPa

TABLE 9.4 : The stresses, due to the permanent and asymmetric load acting at the vault

Due to the deformations and the chosen length of the elements the bending moments calculated with the finite element analyses for the dead load are a bit larger than the moments defined with the analysis. For the asymmetrical load the results of the finite element program calculation match well with the analysis. Due to the asymmetrical load the vault is subjected to huge bending moments and shear forces.

§ 9.5 Vault strengthened with two diagonals running from the crown to the supports.

For a roof accessible to the public the live load is much larger than the variable load acting on a roof subjected to rain and snow only. The life load can act symmetrical as well as asymmetrically. To decrease the bending moments due to the asymmetrical loads the vault is strengthened with the ties running diagonally from the crown to the supports. The prefab structure is composed of two parts connected with a hinge at the top. The curve is described using polar coordinates. The angle between the vertical and the radius pointing to the supports is equal to 2 β. The diagonals are running from the top to the supports. The inclination of these diagonals is equal to β. The span and rise of the arch is equal to respectively 2 a and f. The strengthened vault is statically indeterminate; the distribution of the loads is effected by the stiffness of the vault and the truss. To analyse the transfer of the loads the deformations of the truss and vault are defined for a concentrated load, an equally distributed load and an anti-metrical load.

The deformation of the truss composed of two diagonals

The displacement of the truss, subjected to a vertical load F acting at the top, follows from:

$$\Delta_{TF} = \frac{F\,R}{AE \sin\beta} \qquad [9.40]$$

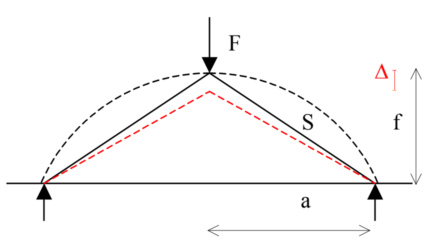

FIGURE 9.12 Deformation of the trussed frame composed of two diagonals due to a vertical force F acting at the top

The deformation of the truss, subjected to a horizontal load H acting at the top, follows from:

$$\Delta_{TH} = \frac{H\,R\tan\beta}{AE\cos\beta} \qquad [9.41]$$

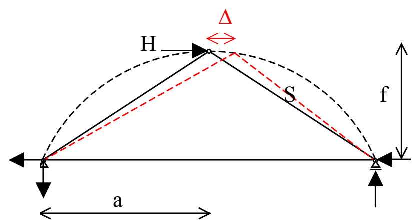

FIGURE 9.13 Deformation of the trussed frame composed of two diagonals due to a horizontal force H acting at the top

The deformation of the vault at the crown

The deformation of the vault at the crown is defined according to the Theory of Maxwell-Mohr with:

$$\Delta = \int\frac{M'\,M\,ds}{EI} + \int\frac{N'\,N\,ds}{EA} \qquad [9.42]$$

With:
M' is the bending moment due to a force F = 1 acting at the top parallel to the deformation;
M = the bending moment due to the load;
N' = the normal force due to a force F = 1 acting at the top parallel to the deformation;
N = the normal force due to the load.

For the symmetrical vault with $\phi \le 2\beta$ the deformation of the vault at the top follows from:

$$\Delta_H = 2\times\int_0^{2\beta}\frac{M\,M'\,R\,d\phi}{EI} + 2\times\int_0^{2\beta}\frac{N\,N'\,R\,d\phi}{EA} \qquad [9.43]$$

Often the maximum bending moments are found for $\phi_u = \beta$ then to simplify the integration the bending moments can be approached with: $M_\phi = M_{max}\sin(\tfrac{1}{2}\pi\,\phi/\beta)$
Substituting M_ϕ into [9.43] gives:

$$\Delta_q = \frac{4\,M'_{max}M_{max}\,R}{EI}\times\int_0^{\beta}\sin^2(\tfrac{1}{2}\pi\,\phi/\beta)\,d(\tfrac{1}{2}\pi\,\phi/\beta)\times\frac{2\beta}{\pi} + \int_0^{2\beta}\frac{N'\,N\,R\,d\phi}{EA} \quad\rightarrow$$

$$\Delta_q = \frac{2\,M'_{max}M_{max}\,R\,\beta}{EI} + \int_0^{2\beta}\frac{N'\,N\,R\,d\phi}{EA} \qquad [9.43']$$

The deformation of the vault subjected by a vertical concentrated force acting at the top

For a concentrated vertical force F acting at the top the bending moment follows from (9.26):

$$M_\phi = \frac{½ F R (1 - \cos\phi)}{\tan\beta} - ½ F R \sin\phi$$

For a vault the normal force is equal to:

$$N_\phi = H_\phi \cos\phi + V_\phi \sin\phi \quad \rightarrow \quad N_\phi = H_\phi/\cos\phi \times (1 + V_\phi \tan\phi/\cos\phi)$$

For a low rise vault the shear force is much smaller than the thrust, then the normal force is approximately equal to $N_f = H_f/\cos\phi$. Substituting the thrust H into this expression gives:

$$N_\phi = \frac{½ F}{\tan\beta \cos\phi}$$

For a concentrated vertical force F = 1 acting at the top the bending moment follows from [9.26]:

$$M'_\phi = \frac{½ R (1 - \cos\phi)}{\tan\beta} - ½ R \sin\phi$$

The normal force acting at the vault is equal to: $N'_\phi = \frac{½}{\tan\beta \cos\phi}$

Substituting M'_ϕ, M_ϕ, N'_ϕ and N_ϕ into [9.43] gives:

$$\Delta_F = \frac{2 F R^2}{4 EI} \times \int_0^{2\beta} \left(\frac{1 - \cos\phi}{\tan\beta} - \sin\phi \right)^2 R\, d\phi + \frac{2 F}{4 AE \tan^2\beta} \times \int_0^{2\beta} \frac{R\, d\phi}{\cos^2\phi}$$

The bending moments are at maximum for $\phi_u = \beta$. Using the symmetry this expression can be simplified:

$$\Delta_F = \frac{F R^3}{EI} \times \int_0^{\beta} \left[\frac{1 + \cos^2\phi - 2\cos\phi}{\tan^2\beta} + \sin^2\phi - \frac{2 (1 - \cos^2\phi) \sin\phi\, d\phi}{\tan\beta} + \frac{½ F R}{AE \tan^2\beta} \times \int_0^{2\beta} \frac{d\phi}{\cos^2\phi}\right.$$

Integrating this expression gives:

$$\Delta_F = \frac{F R^3}{2 EI} \left[\frac{3\beta}{\tan^2\beta} + \beta - \frac{3}{\tan\beta} \right] + \frac{F R}{AE \tan\beta} \qquad [9.44]$$

Assuming the truss is subjected to a vertical force α F and the vault is subjected to a force $(1 - \alpha)$ F. At the top the displacement of the vault is equal to the displacement of the truss, the factor α follows from:

$$\Delta_F (1 - \alpha) = \alpha \Delta_{TF} \quad \rightarrow \quad \alpha = \Delta_F/(\Delta_F + \Delta_{TF}) \qquad [9.45]$$

Example

For the vault described previously with a span l = 2.a = 14.4 m, a rise f = 1.8 m, tan β = 0.25, I = 92.917 × 10^6 mm^4, A = 50 × 10^3 mm^2. Assuming Young's modulus is equal to E = 6750 MPa. The trust is composed of two tubes Ø100-4, running diagonally from the supports to the crown. The area and Young's modulus of the diagonals are respectively A = 1206 mm^2 and 2×10^5 MPa. The deformation is defined for the strengthened vault, subjected to a vertical force F = 1 kN.

The deformation of the truss due to a force F = 1 kN is according to [9.40] equal to:

$$\Delta_{TF} = \frac{10^3 \times 15300}{1206\times2\times10^5 \times\sin\beta} = 0.26 \text{ mm}$$

The deformation of the vault due to the concentrated force F = 1.0 kN is according to [9.44] equal to:

$$\Delta_F = \frac{10^3 \times 15300^3}{2.917\times10^{10}\times6750} \times \left[\frac{3\,\beta^2}{0.25^2} + \beta - \frac{3}{0.25} \right] + \frac{F \times 15300}{50\times10^3\times6750\times0.25} = 11.29 + 0.19 \text{ mm}$$

Substituting the deformation of the truss Δ_{TF} and vault Δ_F into [9.45]:

$$\alpha = \frac{11.29 + 0.19}{11.29 + 0.19 + 0.26} = 0.98$$

The force acting at the truss is equal to F = 0.98 kN. The diagonals are subjected to a force S = 2.02 kN. So the better part of the concentrated load F acting at the top of the vault is transferred by the truss.

The displacement of the vault subjected by a horizontal concentrated force acting at the top

For a horizontal force H acting at the top the bending moment is equal to:

$$M_\phi = \tfrac{1}{2} H R\,(1 - \cos\phi - \tan\beta \sin\phi) \qquad [9.28]$$

The reaction force acting at the supports are equal to ½ H and V = ½ H tan β.The bending moment is maximum for $\phi_u = \beta$. For this load the bending moment is at maximum equal to:

$$M_{max} = \tfrac{1}{2} H R\,(\cos\beta - 1)/\cos\beta \qquad [9.29]$$

For a low rise vault the normal force is approximately equal to $N_\phi = \tfrac{1}{2} H/\cos\phi$

For a horizontal force H = 1 the bending moment and normal force is respectively equal to:

$$M'_{max} = \tfrac{1}{2} R\,(\cos\beta - 1)/\cos\beta$$

For a horizontal force H = 1 the normal force acting at the vault is equal to: $N'_\phi = \tfrac{1}{2}/\cos\phi$

The deformation of the vault at the top is defined according to the Theory of Maxwell-Mohr with expression [9.43]. Substituting M_ϕ into [9.43] gives:

$$\Delta_H = \frac{2 H R^2}{4 EI} \times \int_0^{2\beta} (1 - \cos\phi - \tan\beta \sin\phi)^2 R\, d\phi + \frac{2 H}{4 EA} \times \int_0^{2\beta} \frac{R\, d\phi}{\cos^2\phi}$$

The maximum bending moments are found for $\phi_\upsilon = \beta$. To simplify the integration the bending moments are approached with:$M_\phi = M_{max} \sin(\tfrac{1}{2}\pi\,\phi/\beta)$ The deformation of the vault is approached with [9.43']:

$$\Delta_H = \frac{2 M'_{max} M_{max} R\,\beta}{EI} + \int_0^{2\beta} \frac{N'\, N\, R\, d\phi}{EA} \qquad [9.43']$$

Substituting M_{max} and N_ϕ into expression [9.43'] gives:

$$\Delta_H = \frac{H R^3 \beta}{2 EI} \times \frac{(\cos\beta - 1)^2}{\cos^2\beta} + \frac{H}{2 EA} \times \int_0^{2b} \frac{R\, d\phi}{\cos^2\phi}$$

Integrating this expression gives: $$\Delta_H = \frac{H R^3 \beta\,(\cos\beta - 1)^2}{2 EI \cos^2\beta} + \frac{2 H R \sin\beta \cos\beta}{2 EA\,(1 - 2\sin^2\beta)} \qquad [9.46]$$

Assuming the truss is subjected to a vertical force a F acting downward and the vault is subjected to a force (1-a) F acting upward. At the top the displacement of the vault is equal to the displacement of the truss, the factor α follows from [9.47]:

$$\Delta_H (1 - \alpha) = \alpha \, \Delta_{TH} \quad \rightarrow \quad \alpha = \Delta_H / (\Delta_H + \Delta_{TH}) \qquad [9.47]$$

Example

For the vault described previously the deformation is defined for the vault, subjected to a horizontal force H = 1.0 kN.

The deformation of the truss due to a force H = 1.0 kN is according to [9.42] equal to:

$$\Delta_{TH} = \frac{10^3 \times 15300 \times \tan\beta}{1206 \times 2 \times 10^5 \times \cos\beta} = 0.02 \text{ mm}$$

The deformation of the vault due to the concentrated force H = 1.0 kN is equal to:

$$\Delta_H = \frac{10^3 \times 15.300^3 \times 10^9}{2 \times 92.917 \times 10^6 \times 6750} \times \beta \times \frac{(\cos\beta - 1)^2}{\cos^2\beta} + \frac{10^3 \times 15300^3 \sin\beta \cos\beta}{6750 \times 50 \times 10^3 \times (1 - 2\sin^2\beta)} = 0.66 + 0.01 \text{ mm}$$

Substiuting the deformation of the truss and vault into [9.47] gives:

$$\alpha = \frac{0.66 + 0.01}{0.66 + 0.01 + 0.02} = 0.97$$

Thus the force acting at the truss is equal to a H = 0.97 kN. The force acting at the diagonals is equal to S = 0.5 kN. The better part of the concentrated horizontal force is transfered by the truss.

Equally distributed load

For a vault subjected to an equally distributed load the bending moment follows from [9.11];

$$M_\phi = q\, R^2 \cos^2\beta \, [1 - \cos\phi] - \tfrac{1}{2} q\, R^2 \sin^2\phi \qquad [9.11]$$

The maximum moment acting at the vault is found for $\cos\phi = \cos^2\beta$. Substituting $\cos\phi = \cos^2\beta$ and $\sin^2\phi = (1 - \cos^4\beta)$ into expression [9.11] gives:

$$M_{max} = -\tfrac{1}{2} q\, R^2 \sin^4\beta \qquad [9.12]$$

For a low rise vault the normal force is approximately equal to $N_\phi = H/\cos\phi$, Substituting $H = \tfrac{1}{2} q\, a^2/f$, $a = 2\, R \sin\beta \cos\beta$ and $f = 2\, R \sin^2\beta$ gives:

$$N_\phi = \frac{\tfrac{1}{2} q\, (2\, R \sin\beta \cos\beta)^2}{2\, R \sin^2\beta \cos\phi} \quad \rightarrow \quad N_\phi = \frac{\tfrac{1}{2} q\, R \cos^2\beta}{\cos\phi}$$

For a concentrated vertical force F = 1 acting at the top the bending moment follows from [9.26]:

$$M'_\phi = \frac{\tfrac{1}{2} R\, (1 - \cos\phi)}{\tan\beta} - \tfrac{1}{2} R \sin\phi$$

The normal force acting at the vault is equal to: $N'_\phi = \dfrac{\tfrac{1}{2}}{\tan\beta \cos\phi}$

The deformation of the vault at the top is defined with the Theory of Maxwell/Mohr with (9.43'). To simplify the integration the bending moments are approximated with:

$$M_\phi = M_{max} \sin(\tfrac{1}{2} \pi \, \phi/\beta)$$

Substituting M_ϕ into [9.43] gives:

$$\Delta_q = \frac{2\,M'_{max} M_{max}\, R\,\beta}{EI} + \int_0^{2\beta} \frac{2\,N'\,N\,R\,d\phi}{EA} \qquad [9.43']$$

M'_{max} and M_{max} are defined with respectively expression [9.27] and [9.12]:

$$M'_{max} = -\frac{\tfrac{1}{2}\,R\,(1-\cos\beta)}{\sin\beta} \quad \text{and} \quad M_{max} = -\tfrac{1}{2}\,q\,R^2 \sin^4\beta$$

Substituting M'_{max}, M_{max}, N'_f and N_ϕ into [9.43'] gives:

$$\Delta_q = \frac{q\,R^4 (1-\cos\beta)\sin^3\beta \times \beta}{2\,EI} + \frac{q\,R\cos^2\beta}{EA\tan\beta} \int_0^{2\beta} \frac{R\,d\phi}{\cos^2\phi}$$

Integrating this expression gives:

$$\Delta_q = \frac{q\,R^4 (1-\cos\beta)\sin^3\beta \times \beta}{2\,EI} + \frac{2\,q\,R^2\cos^2\beta}{EA\,(1-\tan^2\beta)} \qquad [9.48]$$

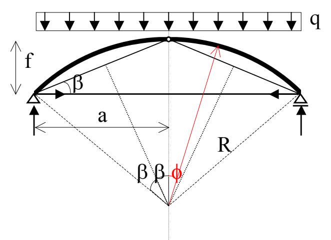

FIGURE 9.14 : Vault strengthened with two diagonals, subjected to an equally distributed load

Assuming the truss is subjected to a vertical force α F acting downward and the vault is subjected to a force a F acting in the opposite direction. At the top the displacement of the vault is equal to the displacement of the truss. The factor α follows from:

$$\Delta_q - \alpha\,\Delta_F = \alpha\,\Delta_{TF} \quad \rightarrow \quad \alpha = \Delta_q/(\Delta_F + \Delta_{TF}) \qquad [9.49]$$

Example

For the vault, described previously the deformation is defined for q = 1 kN/m. Due to a vertical force F = 1.0 kN the deformation of the truss is equal to:

$$\Delta_{TF} = \frac{10^3 \times 15300}{1206 \times 2 \times 10^5 \times \sin^2\beta\cos\beta} = 0.26 \text{ mm}$$

Due to the equally distributed load q = 1.0 kN/m the deformation of the vault is according to [9.48] equal to:

$$\Delta_q = \frac{1.0 \times 15300^4 \times (1-\cos\beta)\sin^3\beta \times \beta}{2 \times 6750 \times 92.917 \times 10^6} + \frac{1.0 \times 15300^2 \times 2\cos^2\beta}{6750 \times 50 \times 10^3 (1-\tan^2\beta)} = 4.56 + 1.39 \text{ mm}$$

Assuming the truss is subjected to a vertical force a F acting downward and the vault is subjected to a force a F acting upward. The factor a follows from [9.49]: $\alpha = \Delta_q/(\Delta_F + \Delta_{TF})$.

Substituting Δ_q = 4.56 + 1.39 mm, Δ_F = 11.29 + 0.18 mm and Δ_{TF} = 0.26 mm:

$$\alpha = \frac{4.56 + 1.39}{11.29 + 0.18 + 0.26} = 0.51$$

Thus the force acting at the truss is equal to a.F = 0.51 kN. The force acting at the diagonals is equal to S = 1.05 kN. The better part of the equally distributed load is transfered by the vault.

For the not strengthened vault the bending moment for $\phi = \beta$ follows from [9.11']:

$$M_{\phi=\beta} = -½\, q\, R^2 (2 \cos^3\beta + 1 - 3 \cos^2\beta)$$

Substituting β, R and q = 1.0 kN/m into [9.11']gives: $M_{\phi=\beta}$ = -0.31 kNm

Due to the upward force the vault is subjected to a maximum bending moment following from [9.27]:

$$M_{F\,max} = ½\, \alpha\, F\, R\, (1-\cos\beta)/\sin\beta$$

Substituting β, R and a F = 0.51 kN/m into [9.12] gives: $M_{\Phi\,max}$ = 0.48 kNm

Due to the strengthening the bending moment acting at the vault is for $\phi = \beta$ equal to:

$$M_{\phi=\beta} = 0.48 - 0.31 = 0.17 \text{ kNm.}$$

Equally distributed surface load

For a vault subjected to an equally distributed surface load the bending moment follows from [9.15]:

$$\frac{M_\phi}{q R^2} = \frac{2\beta\,[1 - \cos\phi]}{\tan\beta} - \phi \sin\phi \qquad [9.15]$$

For low rise vaults the maximum bending moment is approximately found for $\phi = \beta$, then the bending moment is equal to:

$$\frac{M_{\phi=\beta}}{q R^2} = -\frac{\beta\,(1 - 2\cos\beta + \cos^2\beta)}{\sin\beta} \qquad [9.50]$$

For a low rise vault the normal force follows approximately from $N_\phi = H/\cos\phi$. The thrust follows from [9.14]: $H = q\, R\, (2\,\beta/\tan\beta - 1)$.

Substituting H gives: $$N_\phi = \frac{q\, R\, (2\,\beta/\tan\beta - 1)}{\cos\phi}$$

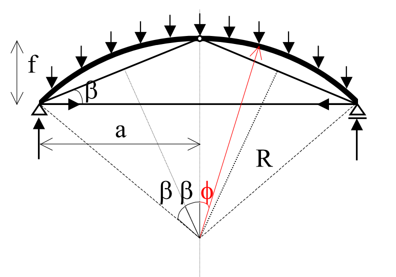

FIGURE 9.15 Vault strengthened with two diagonals, subjected to an equally distributed surface load

For a concentrated vertical force F = 1 acting at the top the bending moment follows from [9.26]:

$$M'_\phi = \frac{\tfrac{1}{2} R (1 - \cos\phi)}{\tan\beta} - \tfrac{1}{2} R \sin\phi$$

The normal force acting at the vault is equal to: $N'_\phi = \dfrac{\tfrac{1}{2}}{\tan\beta \cos\phi}$

The deformation of the vault at the top is defined with the Theory of Maxwell/Mohr with [9.43]. To simplify the integration the bending moments are approached with: $M_\phi = M_{max} \sin(\tfrac{1}{2}\pi\,\phi/\beta)$
Substituting M'_{max} and M_{max}, N'_ϕ and N_ϕ into [9.43'] gives:

$$\Delta_q = \frac{q R^4 (1-\cos\beta)\,\beta^2\,[2\cos\beta - \cos^2\beta - 1]}{EI \sin^2\beta} + \frac{q R\,(2\beta/\tan\beta - 1)}{EA \tan\beta} \times \int_0^{2\beta} \frac{R\,d\phi}{\cos^2\phi}$$

Integrating this expression gives:

$$\Delta_q = \frac{q R^4 \beta^2\,[\,2\cos\beta - \cos^2\beta - 1\,]}{EI\,(1+\cos\beta)} + \frac{q R^2\,(2\beta/\tan\beta - 1)\,2\cos^2\beta}{EA\,(1 - 2\sin^2\beta)} \qquad [9.51]$$

The deformation of the truss has to be equal to the deformation of the vault. Assuming the truss is subjected to a vertical force α F and the vault is subjected to a force α F. the factor a follows from:

$$\Delta_q - \alpha\,\Delta_F = \alpha\,\Delta_{TF} \quad \rightarrow \quad \alpha = \Delta_q/(\Delta_F + \Delta_{TF}) \qquad [9.49]$$

Example

For the vault, described previously the deformation is defined for q = 1.0 kN/m. Due to a vertical force F = 1.0 kN the deformation of the truss is equal to:

$$\Delta_{TF} = \frac{10^3 \times 15300}{1206 \times 2 \times 10^5 \times \sin^2\beta \cos\beta} = 0.26 \text{ mm}$$

Due to the equally distributed surface load q = 1.0 kN/m the deformation of the vault is according to [9.51] equal to:

$$\Delta_q = \frac{1.0 \times 15300^4 \times \beta^2 \times [2\cos\beta - \cos^2\beta - 1]}{6750 \times 92.917 \times 10^6 \times (1+\cos\beta)} + \frac{1.0 \times 15300^2 \times (2\beta/\tan\beta - 1) \times 2\cos^2\beta}{6750 \times 50 \times 10^3 \times (1 - 2.\sin^2\beta)}$$

$$\Delta_q = 2.37 + 1.42 \text{ mm}$$

Assuming the truss is subjected to a vertical force a F and the vault is subjected to a force a F. The factor α follows from [9.49]:

$$\alpha = \Delta_q/(\Delta_F + \Delta_{TF}) \qquad [9.49]$$

Substituting Δ_q = 2.37 + 1.42 mm, Δ_F = 11.29 + 0.18 mm and Δ_{TF} = 0.26 mm:

$$\alpha = \frac{2.37 + 1.42}{11.29 + 0.19 + 0.26} = 0.32$$

Due to the equally distributed surface load acting at the vault the truss is subjected to a force α F = 0.32 kN. The force acting at the diagonal is equal to 0.66 kN. For $\phi = \beta$ the bending moment for the not-strengthened vault follows from [9.23']:

$$M_{max} = -\frac{q R^2 \beta\,(1 - 2\cos\beta + \cos^2\beta)}{\sin\beta}$$

Substituting β, R and q = 1.0 kN/m into (9.24) gives: $M_{\phi=\beta}$ = -0.21 kNm

Due to the upward force α F the vault is subjected to a maximum bending moment following from [9.27]:

$$M_{F\,max} = ½\, \alpha\, F\, R\, (1\text{-}\cos\beta)/\sin\beta$$

Substituting β, R and α F = 0.32 kN/m into [9.12] gives: $M_{F\,max}$ = 0.3 kNm

Due to the strengthening for $\phi = \beta$ the bending moment acting at the vault is equal to:

$$M_{\phi=\beta} = 0.3 - 0.21 = 0.09 \text{ kNm.}$$

Thus for $\phi = \beta$ the bending moment is very small.

Deformation of the vault due to an anti-metrical load

Due to an anti-metrical load the vault will deform horizontally. The deformation of the truss subjected to a horizontal load H acting at the top follows from:

$$\Delta_H = \frac{H\, R \tan\beta}{AE \cos\beta} \qquad [9.41]$$

For a vault subjected to an anti-metrical load the bending moment follows from [9.23]:

$$M_\phi = ½\, q\, R^2 [2 \sin\beta \cos\beta \sin\phi - \sin^2\phi]$$

[9.23]

For $\phi = \beta$ the bending moment is equal to:

$$M_{\phi=\beta} = ½\, q\, R^2 \sin^2\beta\, (2\cos\beta - 1)$$

[9.52]

The shear force V_ϕ is equal to: $V_\phi = q\, R\, (\sin\beta \cos\beta - \sin\phi)$

The normal force follows from: $N_\phi = V_\phi \sin\phi$.

Substituting V_ϕ gives: $N_\phi = q\, R\, (\sin\beta \cos\beta \sin\phi - \sin^2\phi)$

For the vault subjected to a horizontal force H = 1 the bending moment follows from [9.28]

$$M'_\phi = ½\, R\, (1 - \cos\phi - \tan\phi \sin\beta) \qquad [9.28]$$

The bending moment is at maximum for $\phi_u = \beta$. Substituting $\phi_u = \beta$ into expression [9.28] gives the maximum bending moment for H = 1:

$$M'_{max} = ½\, R\, (\cos\beta - 1)/\cos\beta \qquad [9.29]$$

The normal force acting at the vault is equal to: $N'_\phi = ½\, (\cos\phi + \tan\beta \sin\phi)$

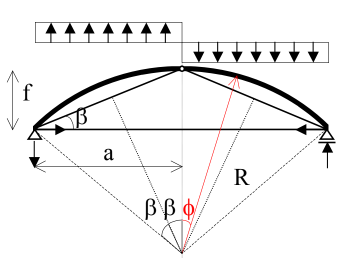

FIGURE 9.16 Vault subjected to an antimetrical load

The deformation of the vault at the top is defined with the Theory of Maxwell/Mohr with [9.43']:

$$\Delta = \frac{2\,M'_{max} M_{max} R\,\beta}{EI} + \frac{2\,N'\,N}{EA}\int_0^{2\beta} R d\phi$$

Substituting M'_{max}, M_{max}, N'_f and N_f into this expression gives:

$$\Delta_q = \frac{\beta\,q\,R^4 \sin^2\beta\,(2\cos^2\beta-1)}{2\,EI} \times \frac{(\cos\beta-1)}{\cos\beta} + \frac{q\,R^2}{EA}\int_0^{2\beta}\underline{(\sin\beta\cos\beta-\sin\phi)\times(\sin\phi\cos\phi+\tan\beta\sin^2\phi)\,d\phi}$$

Integrating this expression gives [9.53]:

$$\Delta_q = \frac{\beta\,q\,R^4\sin^2\beta\,(2\cos^2\beta+1-3\cos\beta)}{2\,EI\cos\beta} + \frac{q\,R^2}{EA}\left[\beta\sin^2\beta + \frac{\sin^3\beta\cos\beta\cos 2\beta}{3} + \frac{2\tan\beta\,(1-\cos 2\beta)}{3}\right] \qquad [9.53]$$

The deformation of the vault subjected to a horizontal force H = 1.0 kN acting at the top follows from [9.46]:

$$\Delta_H = \frac{H\,R^3\beta\,(\cos\beta-1)^2}{2\,EI\cos^2\beta} + \frac{H\,R\,2\sin\beta\cos\beta}{2\,EA\,(1-2\sin^2\beta)} \qquad [9.46]$$

The deformation of the truss has to be equal to the deformation of the vault. Assuming the truss and vault are subjected to a horizontal force α H. The factor α follows from:

$$\Delta_q - \alpha\,\Delta_F = \alpha\,\Delta_{TF} \quad \rightarrow \quad \alpha = \Delta_q/(\Delta_F + \Delta_{TF}) \qquad [9.54]$$

Example

For the vault, described previously, the deformation is defined for an antimetrical load q = 1.0 kN/m. The deformation of the vault, due to the anti-metrical load q = 1.0 kN/m, follows from [9.53]:

$$\Delta_q = \frac{\beta\,q\,R^4\sin^2\beta\,(2\cos\beta+1-3\cos\beta)}{2\times 6750\times 92.917\times 10^6\times\cos\beta} + \frac{15300^2}{6750\times 50\times 10^3}\times\left[\beta\sin^2\beta + \frac{\sin^3\beta\cos\beta\cos 2\beta}{3} + \frac{2\tan\beta\,(1-\cos 2\beta)}{3}\right]$$

$$\Delta_q = 18.22 + 0.03 \text{ m}$$

The deformation of the truss due to a horizontal load H = 1.0 kN acting at the top follows from [9.42]:

$$\Delta_{TH} = \frac{10^3\times 15300\times\sin\beta}{1206\times 2\times 10^5\times\cos^2\beta} = 0.02 \text{ mm}$$

The deformation of the vault due to the concentrated load H follows from [9.46]:

$$\Delta_H = \frac{10^3\times 15.300^3\times 10^9\times\beta\,(\cos\beta-1)^2}{2\times 92.917\times 10^6\times' 6750\cos^2\beta} + \frac{10^3\times 15.300\times 2\sin\beta\cos\beta}{6750\times 50\times 10^3\times(1-2\sin^2\beta)} = 0.66 + 0.01 \text{ mm}$$

Assuming the truss and vault are subjected to a horizontal force α H. The factor α follows from [9.54].
Substituting Δ_q = 18.22 + 0.15 mm, Δ_H = 0.66 + 0.01 mm and Δ_{TH} = 0.02 mm into [9.54]:

$$\alpha = \frac{18.22 + 0.03}{0.66 + 0.01 + 0.02} = 26.5$$

Due to the antimetrical load q = 1.0 kN/m the truss is subjected to a force α H = 26.5 kN. The diagonals are subjected to a force S = ½ H/cos β = 13.7 kN. For $\phi = \beta$ the bending moment due to the force S follows from: $M_{\phi=\beta} = S\,R\,(1-\cos\beta)$. Substituting β, S and R gives: M = 6.3 kNm

For $\phi = \beta$ the bending moment acting at the not strengthened vault follows from [9.23]:

$$M_\phi = \frac{1}{2} q R^2 [2 \sin\beta \cos\beta \sin\phi - \sin^2\phi]$$

Substituting β, R and q = 1.0 kN/m into [9.24] gives: $M_{\phi=\beta} = -6.5$ Nm

Due to the strengthening the bending moment is reduced substantially. $M_{\phi=\beta} = 6.3 - 6.5 = -0.2$ kNm

		Permanent load q = 1.0 kN/m	Live load q = 1.0 kN/m	Anti-metrical live load q = 1.0 kN/m
Reaction, loaded side:	V_B =	7.5 kN	7.2 kN	5.4 kN
Thrust:	H =	14.7 kN	14.4 kN	7.2 kN
Diagonals:	S =	- 0.66 kN	- 1.05 kN	± 13.7 kN
Bending moment:	M_ϕ =	0.09 kNm	0.17 kNm	0.2 kNm

TABLE 9.5 Forces and bending moments for a permanent load, a live load and an anti-metrical load equal to q = 1.0 kN/m.

Conclusions

The better part of the load is transfered by the truss if the vault is subjected to a concentrated horizontal or vertical force acting at the top, but for an equally distributed load the better part is transferred by the vault. The strengthening reduces the bending moments much. Especially for the vault subjected to an ant metrical load th effect the bending moments are reduced substantially.

§ 9.6 Example: vault strengthened with 2 diagonals

The vault, described previously, is strengthened with diagonals running from the top to the supports to reduce the buckling length and the bending moments. Firstly the forces and bending moments are calculated with the described analysis; next the analysis is validated with a computer program. Again the prefabricated vault is composed of circular segments. The span is equal to l = 14.4 m. The rise is of the swallow vault is equal to $f = l/8 = 1.8$ m. The radius R is equal to R = 15.3 m. The angle β is equal to: $\beta = 14.036^0$ .The vault is constructed with a rectangular section with a height of 110 mm. The cardboard tubes ∅60 are positioned with a centre-to-centre distance of 90 mm perpendicular to the span. The vault is subjected to a dead load p_g =3.2 kN/m² and a live load p_e =5.0 kN/m² . For the section with a width of 1.0 m the area and second moment of the area are:

Volume:	V_c =	$110 \times 1000 - 11.1 \times \pi \times 30^2$ =	78.6×10^3	mm²
Area:	A_c =	$(110 - 60) \times 1000$ =	50×10^3	mm²
Second moment of the Area:	I_c =	$1000 \times 110^3/12 - 1000 \times 60^3/12$ =	92.917×10^6	mm⁴

The forces and bending moment due to the permanent load and live load are calculated by multiplying the values found for a load q = 1.0 kN/m. The following table shows the results.

		Permanent load q_g = 3.2 kN/m	Live load q_e = 5.0 kN/m	Anti metrical live load q = 2.5 kN/m_e
Reaction, loaded side:	V_B =	24.0 kN	36.0 kN	13.5 kN
Thrust:	H =	47.0 kN	72.0 kN	18.0 kN
diagonals:	S =	-2.1 kN	-5.3 kN	± 34.3 kN
max. bending moment:	M_ϕ =	0.29 kNm	0.85 kNm	0.5 kNm

TABLE 9.6 Forces and bending moments acting at the vault for q_g = 3.2 kN/m and q_e = 5.0 kN/m .

Vault subjected to an asymmetric equally distributed load

Due to an asymmetric load the vault is subjected to bending. The assumption is made that the live load q is equally distributed, acting on the right side of the vault. To define the forces and bending moments thehe asymmetrical load q cis considered as the combination of an equally distributed load equal to q' = ½ q and an ant metrical load equal to q' =½ q, with q = 5.0 kN/m.

The vertical and horizontal reaction force acting on the supports are respectively:

$V_A = \frac{1}{4} q R \sin(2\beta) = 18$ kN and $V_B = \frac{3}{4} q R \sin(2.\beta) = 54$ kN

The thrust f ollows from [9.17]: $H = \frac{1}{2} q R \cos^2\beta = 0.47 q R = 36$ kN

Equally distributed load

For an equally distributed load q = 1.0 kN/m the force S was defined before: S = 1.05 kN. Thus for q' = 2.5 kN/m the force S acting at the diagonal is equal to: S = 1.05 × 2.5/1.0 = 2.6 kN.

For the non-strengthened vault the bending moment for $\phi = \beta$ follows from [9.11']:

$M_{q\,\phi=\beta} = -\frac{1}{2} q R^2 [2\cos^3\beta + 1 - 3\cos^2\beta]$

Substituting β, R and q' = 2.5 kN/m into [9.11] gives: $M_{q\,\phi=\beta}$ = -0.69 kNm.
Due to the upward force a F the vault is subjected to a maximum bending moment following from [9.27]:

$M_{F\,max} = \frac{1}{2} \alpha F R (1-\cos\beta)/\sin\beta$

Substituting β, R and α F = 0.51 × 2.5 kN/m into [9.27] gives: $M_{F\,max}$ = 1.2 kNm

Due to the strengthening the bending moment is reduced with:

$M_{\phi=\beta} = M_{q\,\phi=\beta} + M_{F\,max} = -0.69 + 1.2$ kNm = 0.51 kNmAnti-metrical load

Due to the anti-metrical load q = 1.0 kN/m the truss is subjected to a force α H = 26.5 kN. The diagonals are subjected to a force S = ± ½ α H/cos β = ± 13.7 kN. For an anti-metrical load equal to q' = 2.5 kN/m the forces acting at the diagonals are equal to: S = ± 34.3 kN.

Due to the force S acting at the diagonals the bending moment is equal to: $M_{\phi=\beta} = S R (1 - \cos\beta)$
Substituting S, R and β gives for the anti-metrical load:

$M_{\phi=\beta} = 34.3 \times 15.3 \times (1 - \cos\beta) = 15.7$ kNm.

For the non strengthened vault the maximum bending moment due to the antimetrical load follows from [9.24]. For q' = ½ × 5.0 = 2.5 kN/m the maximum bending moment is equal to:

$$M_{max} = ½\, q\, R^2 \sin^2\beta \cos^2\beta = ½ \times 2.5 \times 15.3^2 \times \sin^2\beta \cos^2\beta = -16.2 \text{ kNm}$$

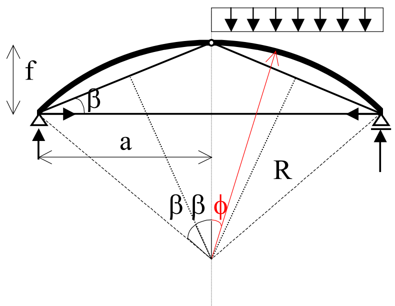

FIGURE 9.17 ault strengthened with two diagonals subjected to an asymmetrical load.Caption

Anti-metrical and equally distributed load

The resulting bending moment due to antimetrical and equally distributed load is equal to:

$$M_{\phi = \beta} = +0.51 \pm (-16.2 + 15.7) = 0.51 \pm 0.5 \text{ kNm}$$

The resulting force S due to the asymmetrical load is equal to: $S = -2.6 \pm 34.3$ kN

Comparing the results for the not strengthened vault and the strengthened vault shows that the strengthening decreases the bending moments substantially. At the unloaded side of the asymmetrical loaded vault the diagonal is compressed, so this element must be dimensioned to resist a normal compressive buckling force.

Analysis of the vault strengthened with a simple truss with computer program

With a computer program, Matrixframe, the forces and bending moments are defined for the vault strengthened with two diagonals running from the crown to the supports.

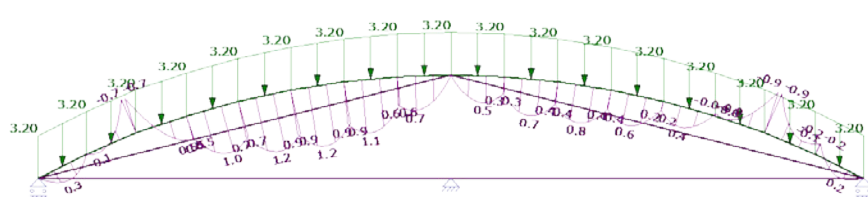

FIGURE 9.18 Bending moments due to the permanent load

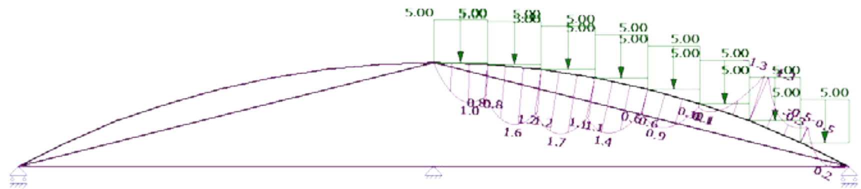

FIGURE 9.19 Bending moments due to the asymmetrical live load

Node	x-coord.	y-coord.	Member	Dead load N	Dead load M	Dead load V	Asym live load N	Asym live load M	Asym live load V
N1	-7.20	0.0	M1:n1-n2	-49.8	0.31	1.34	-0	0	0
N2	-6.360	-0.415	M2: n2-n3	-48.4	0.73	2.16	-0	0	0
N3	-5.497	-0.799	M3; n3-n4	-47.2	0.73	2.71	-0	0	0
N4	-4.612	-1.088	M4: n4-n5	-46.2	0.95	1.72	-0	0	0
N5	-3.711	-1.343	M5: n5-n6	-45.4	1.17	1.65	-0	0	0
N6	-2.795	-1.542	M6: n6-n7	-44.8	1.23	1.43	-0	0	0
N7	-1.869	-1.685	M7: n7-n8	-44.3	1.08	1.80	-0	0	0
N8	-0.937	-1.771	M8: n8-n9	-44.1	0.69	2.11	0	0	0
N9	0	-1.80	M9: n9-n10	-45.5	0.53	1.85	-67.5	1.04	3.23
N10	0.937	-1.771	M10: n10-n11	-45.5	0.73	1.61	-67.9	1.55	2.67
N11	1.869	-1.685	M11: n11-n12	-46.0	0.76	1.55	-68.6	1.65	2.39
N12	2.795	-1.542	M12: n12-n13	-46.6	0.63	1.68	-69.6	1.40	2.70
N13	3.711	-1.343	M13: n13-n14	-47.5	0.41	1.68	-70.8	0.91	2.73
N14	4.612	-1.088	M14: n14-n15	-48.4	0.86	2.27	-72.3	1.29	3.58
N15	5.479	-0.799	M15: n15-n16	-49.6	0.86	2.05	-74.1	1.29	2.87
N16	6.360	-0.415	M16: n16-n21	-51.1	0.25	1.61	-75.9	0.46	2.38
N17	7.20	0	M17 n17-n18	+47.0			+36.0		
N18	0	0	M18: n1-n18	+47.0			+36.0		
			M19: n1-n9	- 3.1			- 37.1		
			M20: n9-n17	- 1.9			+32.4		

TABLE 9.7 Output for a surface load p_g = 3.2 kN/m and the asymmetrical load q_e = 5.0 kN/m

Due to the permanent load q = 3.2 kN/m and variable load q = 5.0 kN/m the bending moment is at maximum for member 11.

N_{rep} = -46.0 – 68.6 = - 114.6 kN $\sigma = N/A =$ -2.29 MPa

M_{rep} = 0.76 + 1.65 = 2.41 kNm $\sigma = M\,z/I =$ 1.43 MPa

The bending tensile stress is smaller than the normal compressive stress, so the stiffness of the vault is not reduced by cracks.

Ultimate state

Due to the permanent load q = 3.2 kN/m and a variable load q = 5.0 kN/m acting at one half of the vault the bending moment is at maximum for node 5.

N_d = - 1.2 × 46.0 – 1.5 × 68.6 = - 158.1 kN, $\sigma_d = N_d/A =$ -3.16 MPa

M_d = 1.2 × 0.76 + 1.5 × 1.65 = 3.39 kNm, $\sigma_d = M_d\,z/I =$ 2.01 MPa

Buckling

For prefabricated structures the quality of the concrete is at least C35/40, with $E_0 = 3.4 \times 10^4$ MPa. The bending stresses are much smaller than the compressive normal stresses, so the structure is not cracked. For the ultimate state the deformations increase by creep. Nevertheless the stiffness of an uncracked section is at least equal to $EI = E_{0,t} I$ with $E_{0,t} = \phi_{cd}/(1.75 \times 10^{-3})$ and $f_{cd} = f_{ck}/1.5$.

For C35/40: $E_{0,t} = (35/1.5)/(1.75 \times 10^{-3}) = 13.3 \times 10^3$ MPa.

The critical buckling force according to Euler is equal to: $N_{cr} = \dfrac{\pi^2 EI}{(\frac{1}{2} s)^2}$

For this structure the length of the vault between the top and support is equal to: $s = 2 \beta R = 7.5$ m

$$N_{cr} = \frac{\pi^2 \times 92.917 \times 10^6 \times 13.3 \times 10^3}{(0.5 \times 7.5)^2 \times 10^6} = 867 \times 10^3 \text{ N} \qquad n_{cr} = N_{cr}/N = 867/158.1 = 5.5$$

The maximum stress follows from: $\sigma = \dfrac{N}{A} \pm \dfrac{M z}{I} \times \dfrac{n_{cr}}{(n_{cr}-1)}$

$$\sigma_d = \frac{-158.1 \times 10^3}{50 \times 10^3} \pm \frac{3.39 \times 10^6 \times 55}{92.917 \times 10^6} \times \frac{5.5}{(5.5-1)} = -3.16 \pm 2.45 \text{ MPa}$$

The bending stress is smaller than the normal stress, the section is not cracked.

Shear stresses

The vault has a thickness h = 110 mm. The radius of the infill tubes is r = 30 mm. The centre to centre distance c of the tubes is 90 mm. The thickness of the flange above or under the tubes is 25 mm. Due to the load the structure is subjected to a shear force: $V_d = dM_d/dx$,

For member 14: $V_d = 1.2 \times 2.27 + 1.5 \times 3.58 = 8.1$ kN.

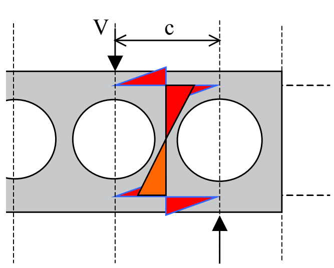

FIGURE 9.20 The transfer of the shear force

The shear stresses acting at the sections above and below the infills follows from:

$$\tau_{dmean} = V_d/(b\,t): \qquad \tau_{dmean} = \frac{8.1 \times 10^3}{1000 \times 2 \times 25} = 0.16$$

Due to the shear force the upper and lower flange is subjected to bending moments. The bending moment M acting at the lower and upper flange is: $M_d = V_d\, c/4$

The bending moment M acting at the strut between two tubes is two times the bending moment acting on the upper and lower flange: $M_d = 2 \times V_d\, c/4 = ½V_d\, c$

Between two tubes the shear force is: $H_d = M_d /(½\, z)$

with: $½\, z = (110-25)/2 = 42.5$

The shear force acting at strut between the tubes follows from:

$$H_d = \frac{\Sigma M_d}{z} = \frac{2\, V_d \times c/4}{½\, z} = V_d\, c/z$$

Substituting z = 110 - 25 = 85 mm and c = 90 mm gives: $H_d = 8.1 \times 90/85 = 8.58$ kN

The shear stress is: $\tau_{dmean} = H_d/(b\,t)$ → $\tau_{dmean} = 8580/(1000 \times 30) = 0.29$ MPa

For concrete the ultimate shear force follows from: $V_u \leq 0.035\, k^{1.5} \sqrt{f_{c.k}} \times b_w d$

With:

$k = 1+\sqrt{(200/d)} \leq 2$,

d = 30 mm, k = 3.6 > 2 → k = 2

$b_w = 1000$ mm

For concrete C35/45 the ultimate compressive stress is: $f_{ck} = 35$ N/mm^2, thus the ultimate shear stress is: $\tau_u \leq 0.035 \times 2^{1.5} \times \sqrt{35} = 0.59$ MPa

Diagonals

Due to the asymmetrical load the diagonal at the unloaded side is subjected to a compressive normal force: $N_d = -1.5 \times 37.1 = 55.7$ kN.

The assumption is made that the diagonal is Æ100´ 4, S235, length 7.42 m, A = 1206 mm^2, I = 1.39 × 10^6 mm^4. The diagonal is supported halfway the top, the buckling length is equal to 3.71 m.

Buckling force: $$N_{cr} = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2.0 \times 10^5 \times 1.39 \times 10^6}{3710^2} = 199 \times 10^3 \text{ N}$$

$$n_{cr} = N_{cr} / N_d = 199/55.7 = 3.6$$

Dead load: q = 0.1 kN/m, $M = 1.2 \times {}^1/_8 \times 0.1 \times 3.7^2 = 2.05$ kNm

Stress: $$\sigma_d = \frac{55700}{1206} + \frac{2.05 \times 10^6 \times 100/2}{1.39 \times 10^6} \times \frac{3.6}{(3.6-1)} = 46 + 102 < 235 \text{ MPa}$$

Conclusions

Due to strengthening the sections of the vault are loaded by relative small bending moments and shear forces. The compressive stresses are larger than the tensile stresses caused by bending, so the structure is not cracked. Due to the strengthening the shear forces and shear stresses are small. The infills do not decrease the structural resistance much. The slender vault can resist the given heavy loads very well.

§ 9.7 Approach to define the forces acting at the diagonals for the strengthened vault

Due to the strengthening the bending moments acting at the vault are very small. For the design of the strengthened vault the forces S acting at the diagonals can be defined with the following simplification by adding virtual hinges for $\phi = \pm \beta$.

Strengthened vault subjected to an equally distributed load q:

The assumption is made that the vault is subjected to an equally distributed load q. The vertical reaction acting at the supports V_A or V_B is equal to: $V_A = V_B = q\,a$. To define the force S acting at these bars virtual hinges are added. The force S acting in the tie has a horizontal component $S \cos\beta$ and a vertical component $S \sin\beta$. The assumption is made that the bending moment is zero for $\phi = \beta$:

$$M_x = H\,y - \tfrac{1}{2} q\,x^2 + (S\cos\beta)\,y - (S\sin\beta)\,x = 0$$

Substituting: $x = R\sin\phi$; $y = R(1-\cos\phi)$; $f = 2R\sin^2\beta$; $a = R\sin(2\beta) = 2R\sin\beta\cos\beta$

$$H = \tfrac{1}{2} q\,a^2/f \quad \rightarrow \quad H = q\,R\cos^2\beta$$

$$M_\phi = q\,R^2\cos^2\beta\,(1-\cos\phi) - \tfrac{1}{2} q\,R^2\sin^2\phi + S\,R\,[\cos\beta\,(1-\cos\phi) - \sin\beta\sin\phi\,]$$

The force S is defined in case a virtual hinge is added at $\phi = \beta$. Thus for $\phi = \beta$ the bending moment is zero: $M_\phi = 0$

$$q\,R^2\cos^2\beta\,(1-\cos\beta) - \tfrac{1}{2} q\,R^2\sin^2\beta = S\,R\,[\sin^2\beta - \cos\beta\,(1-\cos\beta)\,] \rightarrow$$

$$\frac{S}{q\,R} = \frac{\cos^2\beta\,(1-\cos\beta) - \tfrac{1}{2}(1-\cos\beta)(1+\cos\beta)}{(1+\cos\beta)(1-\cos\beta) - \cos\beta\,(1-\cos\beta)} \rightarrow S = q\,R\,[\cos^2\beta - \tfrac{1}{2}\cos\beta - \tfrac{1}{2}] \qquad [9.55]$$

The normal force S is negative, so the diagonal is compressed.

$$\frac{M_\phi}{q\,R^2} = \cos^2\beta\,(1-\cos\phi) - \tfrac{1}{2}\sin^2\phi + [\cos^2\beta - \tfrac{1}{2}\cos\beta - \tfrac{1}{2}] \times [\cos\beta\,(1-\cos\phi) - \sin\beta\sin\phi]$$

Differentiating the expression for the bending moment results in:

$$\frac{dM_\phi}{q\,R^2\,d\phi} = \cos^2\beta\sin\phi - \sin\phi\cos\phi + [\cos^2\beta - \tfrac{1}{2}\cos\beta - \tfrac{1}{2}] \times [\cos\beta\sin\phi - \sin\beta\cos\phi\,]$$

The bending moment is maximum for $dM_\phi/d\phi = 0$, thus:

$$(+\cos^3\beta + \tfrac{1}{2}\cos^2\beta - \tfrac{1}{2}\cos\beta)\sin\phi - \sin\phi\cos\phi - [\cos^2\beta - \tfrac{1}{2}\cos\beta - \tfrac{1}{2}]\sin\beta\cos\phi = 0$$

$$(\cos^2\beta + \tfrac{1}{2}\cos\beta - \tfrac{1}{2})\cos\beta\tan\phi - \sin\phi = [\cos^2\beta - \tfrac{1}{2}\cos\beta - \tfrac{1}{2}]\sin\beta$$

This equation can be solved numerically. Next the bending moment is calculated for this angle ϕ

Vault subjected to an equally distributed surface load q:

Due to the dead load the vault will be subjected to an equally distributed surface load q. Again the three hinged arch is statically determinate. For the three hinged vaulth subjected to a surface load q the vertical reaction acting at the supports V_A and V_B is equal to:

$$V_A = V_B = \int_{\phi=0}^{\phi=2\beta} q\,R\,d\phi = 2\,q\,R\,\beta$$

The moment M_0 at the centre due to the distributed surface load is equal to zero:

$$M_0 = V\,R\sin(2\beta) - H\,f - \int_{\phi=0}^{\phi=2\beta} q\,R\,d\phi\,R\sin\phi = 0$$

Substituting $V = 2\,q\,R\,\beta$ and integrating between the constraints $\phi = 0$ and $\phi = 2\beta$ gives:

$$M_0 = 2\,q\,R^2\beta\sin(2\beta) - H\,f - q\,R^2[1 - \cos(2\beta)] = 0$$

Next the bending moment is divided by the lever arm to define the thrust H:

$$H = \frac{q\,R^2[2\beta\sin(2\beta) + \cos(2\beta) - 1]}{R[1 - \cos(2\alpha)]}$$

Substituting $\cos(2\beta) = 1 - 2\sin^2\beta$ and $\sin 2\beta = 2\sin\beta\cos\beta$ gives:

$$H = \frac{q\,R[2\beta\cos\beta - \sin\beta]}{\sin\beta}$$

The bending moment for an angle ϕ_1 follows from:

$$M_{\phi 1} = H\,R[1-\cos\phi_1] - q\,R^2\int_{\phi=0}^{\phi 1}(\sin\phi_1 - \sin\phi)\,d\phi + S\,R[\cos\beta(1-\cos\phi_1) - \sin\beta\sin\phi_1]$$

Integrating this expression between the constraints $\phi = 0$ and $\phi = \phi_1$ gives:

$$\frac{M_{\phi 1}}{q\,R^2} = \frac{H}{q\,R}(1-\cos\phi_1) - \phi_1\sin\phi_1 - \cos\phi_d\Big|_{\phi=0}^{\phi 1} + \frac{S}{q\,R}[\cos\beta(1-\cos\phi_1) - \sin\beta\sin\phi_1]$$

For $\phi_1 = \phi$ this equation for the bending moment becomes:

$$\frac{M_\phi}{q\,R^2} = \frac{H}{q\,R}(1-\cos\phi) - \phi\sin\phi - \cos\phi + 1 + \frac{S}{q\,R}[\cos\beta(1-\cos\phi) - \sin\beta\sin\phi]$$

Substituting H into the expression gives:

$$\frac{M_\phi}{q\,R^2} = \frac{[2\beta\cos\beta - \sin\beta]}{\sin\beta} \times [1 - \cos\phi] - \phi\sin\phi + [1-\cos\phi] + \frac{S}{q\,R}[\cos\beta(1-\cos\phi) - \sin\beta\sin\phi] \rightarrow$$

$$\frac{M_\phi}{qR^2} = \frac{2\beta\cos\beta}{\sin\beta}[1-\cos\phi] - \phi\sin\phi + \frac{S}{q\,R}[\cos\beta(1-\cos\phi) - \sin\beta\sin\phi]$$

The assumption is made that a hinge is made at $\phi = \beta$. Consequently for $\phi = \beta$ the bending moment is equal to zero:

$$\frac{S}{q\,R} = \frac{2\beta\cos\beta(1-\cos\beta)/\sin\beta - \beta\sin\beta}{\sin^2\beta - \cos\beta(1-\cos\beta)}$$

Multiply the numerator and denominator of this expression with $\sin\beta$:

$$\frac{S}{q\,R} = \frac{2\beta\cos\beta[1-\cos\beta] - \beta(1+\cos\beta)\times(1-\cos\beta)}{\sin\beta[(1+\cos\beta)\times(1-\cos\beta) - \cos\beta(1-\cos\beta)]}$$

$$\frac{S}{q\,R} = \frac{2\beta\cos\beta - \beta(1+\cos\beta)}{\sin\beta[1+\cos\beta - \cos\beta]} \rightarrow \frac{S}{q\,R} = -\frac{\beta[1-\cos\beta]}{\sin\beta} \quad [9.56]$$

The sign is negative, the force S is compressed.

Vault subjected to an asymmetric distributed load

Due to an asymmetric load the vault is subjected to bending. The assumption is made that the live load q is equally distributed, acting on the right side of the arch. The vertical and horizontal reaction force acting on the supports are respectively:

$$V_A = \tfrac{1}{4}\, q\, R \sin(2\beta) \quad \text{and} \quad V_B = \tfrac{3}{4}\, q\, R \sin(2\beta)$$

The thrust is calculated for the left part of the vault with the equilibrium of the bending moment around the top.

$$M_{\phi=0} = H f - \tfrac{1}{4} q a^2 = 0 \quad \rightarrow \quad H = \tfrac{1}{4} q a^2/f \quad \rightarrow \quad H = \tfrac{1}{2} q R \cos^2\beta$$

The thrust can be calculated for the right part of the vault in the same way with the same result:

$$M_{\phi=0} = \tfrac{3}{4} q a^2 - H f - \tfrac{1}{2} q a^2 = 0 \quad \rightarrow \quad H = \tfrac{1}{4} q a^2/f \quad \rightarrow \quad H = \tfrac{1}{2} q R \cos^2\beta$$

For the right side the bending moment M_f is calculated for a certain angle ϕ_1 from the top with:

$$\frac{M_{\phi 1}}{q R^2} = \frac{H}{q R}(1 - \cos\phi) + \tfrac{1}{4}\sin(2\beta)\sin\phi - \tfrac{1}{2}\sin^2\phi + \frac{S}{q R}[\cos\beta\,(1-\cos\phi) - \sin\beta\sin\phi]$$

Substituting H and $\sin(2\beta) = 2\cos\beta\sin\beta$:

$$\frac{M_{\phi 1}}{q R^2} = \tfrac{1}{2}\cos^2\beta\,(1 - \cos\phi) + \tfrac{1}{2}\sin\beta\cos\beta\sin\phi - \tfrac{1}{2}\sin^2\phi + \frac{S}{q R}[\cos\beta\,(1-\cos\phi) - \sin\beta\sin\phi]$$

To define the force S a hinge is introduced for $\phi = \beta$. So for $\phi = \beta$ the bending moment is zero:

$$\frac{S}{q R} = \tfrac{1}{2} \times \frac{[\cos^2\beta\,(1 - \cos\beta) + \sin^2\beta\cos\beta - \sin^2\beta]}{\sin^2\beta - \cos\beta\,(1-\cos\beta)} \quad \rightarrow$$

$$\frac{S}{q R} = \tfrac{1}{2} \times \frac{[\cos^2\beta\,(1-\cos\beta) - \sin^2\beta\,(1 - \cos\beta)]}{(1+\cos\beta)(1-\cos\beta) - \cos\beta\,(1-\cos\beta)}$$

Divide by $[1-\cos\beta]$:

$$S = \tfrac{1}{2} q R\,[\cos^2\beta - \sin^2\beta] \quad \rightarrow \quad S = \tfrac{1}{2} q R \cos(2\beta) \qquad [9.57]$$

Substituting S into M_ϕ:

$$\frac{2 M_\phi}{q R^2} = \cos^2\beta\,(1-\cos\phi) + \sin\beta\cos\beta\sin\phi - \sin^2\phi + \cos(2\beta) \times [\cos\beta\,(1-\cos\phi) - \sin\beta\sin\phi]$$

The bending moment is maximum for $dM_f/d\phi = 0$

$$\cos^2\beta\sin\phi + \tfrac{1}{2}\sin(2\beta)\cos\phi - 2\sin\phi\cos\phi + \cos(2\beta) \times [\cos\beta\sin\phi - \sin\beta\cos\phi] = 0 \quad \rightarrow$$

$$[\cos^2\beta + \cos(2\beta)\cos\beta]\sin\phi - 2\sin\phi\cos\phi - \cos(2\beta)\sin\beta\cos\phi = 0$$

Dividing this expression by $\cos\phi$ gives:

$$[\cos^2\beta + \cos(2\beta)\cos\beta]\tan\phi = 2\sin\phi + \cos(2\beta)\sin\beta$$

To find the maximum moment this equation has to be solved numerically.

Example vault strengthened with 2 diagonals

The vault, described previously, is strengthened with diagonals running from the top to the supports to reduce the buckling length and the bending moments. The forces and bending moments are calculated with the described approach. Again the prefabricated vault is composed of circular segments. The span is equal to l = 14.4 m. The rise is of the swallow vault is equal to f = l/8 = 1.8 m. The radius R is equal to R = 15.3 m. The angle b is equal to: $\beta = 14.036^0$ .The vault is constructed with a rectangular section with a height of 110 mm. The cardboard tubes Ø60 are positioned with a centre to centre distance of 100 mm perpendicular to the span. The structure is subjected to a dead load p_g = 3.2 kN/m^2 and a ive load p_e = 5.0 kN/m^2. For the section with a width of 1.0 m the area and second moment of the area are:

$$A = (110 - 60) \times 1000 = 50 \times 10^3 \text{ mm}^2$$

$$I = 1000 \times 110^3/12 - 1000 \times 60^3/12 = 92.917 \times 10^6 \text{ mm}^4$$

Permanent load:

Due to the permanent load the arch will be subjected to an equally distributed surface load q = 3.2 kN/m. Due to the surface load q the vertical reaction acting at the supports V_A and V_B is equal to:

$$V_A = V_B = \int_{\phi=0}^{\phi=2\beta} q\, R\, d\phi = 2\beta q R = 0.49 \times 3.2 \times 15.3 = 24 \text{ kN}$$

The thrust follows from [9.14]:

$$H = q R \frac{[2\beta \cos\beta - \sin\beta]}{\sin\beta} = \frac{3.2 \times 15.3 \times [0.49 \times \cos 14.036^0 - \sin 14.036^0]}{\sin 14.036^0} = 47 \text{ kN}$$

The force S acting on the diagonal follows from [9.56]:

$$\frac{S}{q R} = \frac{\beta [\cos\beta - 1]}{\sin\beta} = \frac{0.49/2 \times [\cos 14.036 - 1]}{\sin 14.036} = -0.03 \quad \rightarrow \quad S = -1.48 \text{ kN}$$

The bending moment for a certain angle ϕ follows from:

$$\frac{M_\phi}{q R^2} = \frac{2\beta \cos\beta [1 - \cos\phi]}{\sin\beta} - \phi \sin\phi + \frac{S}{q R} [\cos\beta (1 - \cos\phi) - \sin\beta \sin\phi]$$

Substituting the force S and the angle β = 14.036 into this expression gives:

$$\frac{M_\phi}{q R^2} = \frac{0.49 \times \cos 14.036 \times [1 - \cos\phi]}{\sin 14.036} - \phi \sin\phi - 0.03 \times [\cos 14.036 \times (1 - \cos\phi) - \sin 14.036 \times \sin\phi]$$

$$\frac{M_\phi}{q R^2} = 1.93 \times [1 - \cos\phi] - \phi \times \sin\phi + 0.0073 \times \sin\phi$$

The bending moment is maximum for dM = 0, differentiating gives:

$$\frac{dM_\phi}{d\phi} = 1.93 \times \sin\phi - \sin\phi - \phi \cos\phi + 0.0073 \times \cos\phi = 0 \quad \rightarrow \quad 0.93 \times \tan\phi = \phi - 0.0073;$$

This expression has a solution for $\phi = 6.5^0$. Substituting this angle into the expression for the bending moment gives: $M = 0.00039\, q R^2 = 0.29$ kNm

Vault subjected to the equally distributed live load q:

The vertical reaction acting at the supports V_A and V_B is equal to:

$V_A = V_B = q\,a = 5 \times 7.2 = 36$ kN.

The thrust follows from: $H = ½\, q\, a^2/f = q\, R \cos^2\beta \quad \rightarrow \quad H = 72.0$ kN

The force acting at the diagonals follows from [9.55]:

$S = q\, R\, [\cos^2\beta - ½ \cos\beta - ½] \quad \rightarrow \quad S = -3.36$ kN

The bending moment follows from:

$$M_\phi = H\, R\, (1 - \cos\phi) - ½\, q\, R^2 \sin^2\phi + S\, R\, [\cos\beta\, (1 - \cos\phi) - \sin\beta \sin\phi]$$

Substituting H, S and β gives:

$$M_\phi = 72.0 \times 15.3 \times (1 - \cos\phi) - 2.5 \times 15.3^2 \times \sin^2\phi - 3.36 \times 15.3 \times [0.97 \times (1 - \cos\phi) - 0.2425 \times \sin\phi]$$

$$M_\phi = (1102 - 51.4) \times (1 - \cos\phi) - 585 \times \sin^2\phi + 51.4 \times 0.2425 \times \sin\phi$$

The bending moment is at maximum for $dM/d\phi = 0$

$$1050.6 \times \sin\phi - 1170 \times \sin\phi \cos\phi + 12.5 \times \cos\phi = 0$$

This equation can solved numerically. The bending moments are very small:

For $\phi = \beta$	$M = -0.02$ kNm
For $\phi = {}^1/_2\,\beta$	$M = 0.67$ kNm
For $\phi = {}^3/_2\,\beta$	$M = 0.85$ kNm

Vault subjected to an asymmetric equally distributed load

Due to an asymmetric load the vault is subjected to bending. The assumption is made that the live load q is equally distributed, acting on the right side of the vault. The vertical and horizontal reaction force acting on the supports are respectively:

$V_A = ¼\, q\, R \sin(2\beta) = 18$ kN and $V_B = ¾\, q\, R \sin(2\beta) = 54$ kN

The thrust follows from: $H = ½\, q\, R \cos^2\beta = 0.47\, q\, R = 36$ kN

For the right side the force acting on the diagonal follows from [9.57]:

$S = ½\, q\, R \cos(2\beta) = 0.441 \times q\, R = 33.75$ kN

For the right side the bending moment M_ϕ is calculated for a certain angle ϕ from the top with:

$$\frac{M_\phi}{q\, R^2} = \frac{H}{q\, R}(1 - \cos\phi) + ¼ \sin(2\beta) \sin\phi - ½ \sin^2\phi + \frac{S}{q\, R}[\cos\beta\, (1 - \cos\phi) - \sin\beta \sin\phi]$$

Substituting H, S and β into the expression for the bending moment:

$$\frac{M_\phi}{q\, R^2} = 0.47 \times (1 - \cos\phi) + 0.118 \times \sin\phi - ½ \sin^2\phi + 0.441 \times [0.97 \times (1 - \cos\phi) - 0.2425 \times \sin\phi]$$

$$\frac{M_\phi}{q\, R^2} = 0.9 - 0.9 \times \cos\phi + 0.011 \times \sin\phi - ½ \sin^2\phi$$

The bending moment is at maximum for $dM/d\phi = 0$:

$$0.9 \times \sin\phi + 0.011 \times \cos\phi - \sin\phi \cos\phi = 0$$

The angle ϕ is defined by dividing this expression by $\cos\phi$:

$$0.9 \times \tan\phi + 0.011 - \sin\phi = 0 \quad \rightarrow \quad \phi \approx 22^0$$

For this angle the bending moment is: $\frac{M_\phi}{q R^2} = -0.00051$ and $M_\phi = -0.6$ kNm

		Permanent load q =1.0 kN/m	Live load q =1.0 kN/m	Anti -etrical load q =1.0 kN/m
Reaction, loaded side:	V_B =	24.0 kN	36.0 kN	54.0 kN
Thrust:	H =	47.0 kN	72.0 kN	36.0 kN
Diagonals:	S =	- 1.5 kN	- 3.4 kN	-1.7 ± 33.75 kN
Max. bending moment:	M_ϕ =	-0.29 kNm	0.85 kNm	± 0.6 kNm

TABLE 9.8 Forces and bending moments acting at the vault.

Comparing the results with the previous analysis shows that the results of the approach do not differ much from the results calculated before. The approach can be used to estimate forces and bending moments quickly in an early stage of the process of design. In practice the forces acting in the diagonals are effected by the stiffness of the joints. With the described expresions it is possible to study the effect of varying forces S acting at the diagonals.

§ 9.8 Vault strengthened with diagonal cables running from the supports

At the beginning of the 19th century the Russian engineer V.G Shukhof strengthened the half circular aches of the GUM department store in Moscow with tensioned cables [Bel62]. in the same way we can reduce the bending moments for a low rise vault. Two cables are added running form the supports to the vault at a point halfway the crown and supports. The forces and bending moments are calculated with a computer program (Matrixframe).

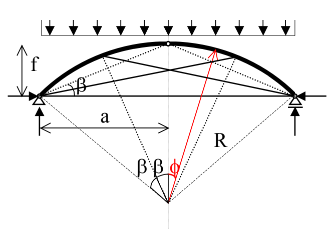

FIGURE 9.21 Vault strengthened with cables running from the supports to the vault

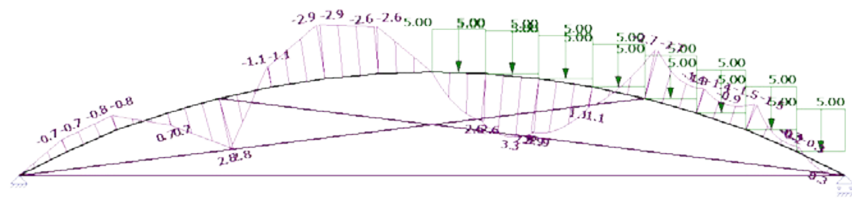

FIGURE 9.22 Bending moments due to the asymmetrical live load

Node	x-coord.	y-coord.	Member	Dead load N	Dead load M	Asym live load N	Asym live load M	Asym live load V
n1	-7.20	0.0	M1:n1-n2	- 52.8	0.45	- 16.6	0.72	0.75
n2	-6.360	-0.415	M2: n2-n3	- 51.5	1.51	- 16.6	0.80	0.09
n3	-5.497	-0.779	M3; n3-n4	- 50.3	1.51	- 16.6	0.80	1.62
n4	-4.612	-1.088	M4: n4-n5	- 49.3	0.34	- 16.5	2.83	2.27
n5	-3.711	-1.343	M5: n5-n6	- 49.8	0.33	- 37.8	2.83	4.22
n6	-2.795	-1.542	M6: n6-n7	- 49.2	0.32	- 38.0	2.93	1.92
n7	-1.869	-1.685	M7: n7-n8	- 48.7	0.34	- 38.0	2.93	0.40
n8	-0.937	-1.771	M8: n8-n9	- 48.5	0.35	- 37.9	2.55	2.72
n9	0	-1.80	M9: n9-n10	- 48.5	0.35	- 37.8	2.55	5.06
n10	0.937	-1.771	M10: n10-n11	- 48.7	0.32	- 38.2	3.28	2.69
n11	1.869	-1.685	M11: n11-n12	- 49.2	0.28	- 38.9	2.91	4.19
n12	2.795	-1.542	M12: n12-n13	- 49.8	0.27	- 40.1	2.69	6.30
n13	3.711	-1.343	M13: n13-n14	- 49.4	0.40	- 59.2	2.69	3.57
n14	4.612	-1.088	M14: n14-n15	- 50.3	1.23	- 60.5	1.46	2.16
n15	5.479	-0.799	M15: n15-n16	- 51.6	0.47	- 62.2	1.46	3.25
n16	6.360	-0.415	M16: n16-n21	- 52.9	0.47	- 64.0	0.31	2.18
n17	7.20	0	M17: n17-n18	+ 1.5		+22.8		
n18	0	-0.886	M18: n1-n18	+ 1.4		- 19.0		
			M20: n5-n18	+ 1.5		+22.8		
			M21: n13-n18	+ 1.4		- 19.0		
			M22: n1-n19	+45.6		+ 34.1		
			M23: n19-n17	+45.6		+ 34.1		

TABLE 9.9 : Bending moments and forces for a surface load p_g = 3.2 kN/m and an asymmetrical load q_e = 5.0 kN/m

Due to the permanent load q = 3.2 kN/m and variable load q = 5.0 kN/m the bending moment is at maximum for member 10.

N_{rep} = -48.7 – 38.2 = - 86.9 kN, σ = N/A = -1.74 MPa

M_{rep} = 0.32 + 3.28 = 3.60 kNm, σ = M.z/I = 2.13 MPa

Post-tensioning

Due to the asymmetrical load the diagonal ties are compressed, F = -19.0 kN. To tension these ties have to be post-tensioned. Due to the post-tensioning of the diagonals the vault is subjected to normal forces and bending moments. A finite element calculation is made for an arificial load $F_{p\,hor}$ = 8.12 kN, $F_{p\,vert}$ = 1.0 kN. For this loads the post-tensioning force is equal to F_p = 8.2 kN.

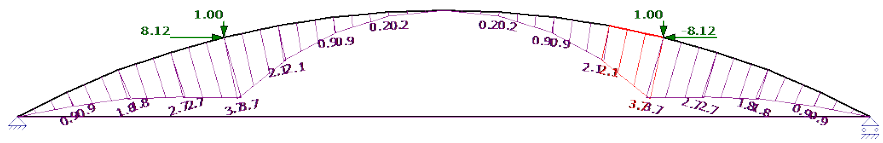

FIGURE 9.23 Bending moments due to a post-tensioning Force equal to F_p = 8.2 kN

The vault is at node n5 and n13 subjected to a maximum bending moment and normal force of respectively: M = 3.65 kNm and a normal force N = 7.8 kN. To compensate the tensile force acting in the tie the post-tensioning must be larger than F = 19.0 − 1.4 = 17.6 kN. Then the bending moment is equal to: M = 3.65 × 17.6/8.2 = 7.8 kNm. This moment is quite large, but smaller than the maximum bending moment for the not-strengthened vault M = 16.95 kNm.

Ultimate state

To compensate the compressive force for the ultimate state the minimal post-tensioning force follows from: $F_p = 1.5 \times 19 - 0.9 \times 1.4 = 27.2$ kN. For this load the normal force and bending moment due to the post-tensioning are in node n13 respectively:

$N_p = 7.8 \times 27.2/8.2 = 25.9$ kN, $\quad M_p = 3.65 \times 27.2/8.2 = 12.1$ kNm.

Due to the permanent load, the post-tensioning and the variable load the normal force and bending moment at member 12 are respectively:

$N_d = -1.2 \times 49.8 - 1.0 \times 25.9 - 1.5 \times 40.1 = -145.8$ kN

$M_d = 1.2 \times 0.27 + 1.0 \times 12.1 + 1.5 \times 2.69 = 16.5$ kNm

The critical buckling force according to Euler is equal to: $N_{cr} = \dfrac{\pi^2 EI}{s_1^2}$

For the strengthened vault the length of the vault between the nodes is equal to: $s_1 = ½\, \beta\, R = 3.75$ m

For prefabricated structures the quality of the concrete is at least C35/45, with $E_0 = 3.4 \times 10^4$ MPa. For the ultimate state the deformations increase by creep. Probably the sections are not cracked, for the design of the structure the stiffness of the vault can be approached with:

$EI = E_{0,t}\, I$ with: $E_{0,t} = f_{cd}/(1.75\times10^{-3})$ and $f_{cd} = f_{ck}/1.5$

For C35/45: $E_{0,t} = (35/1.5)/(1.75\times10^{-3}) = 13.3 \times 10^3$ MPa.

$$N_{cr} = \frac{\pi^2 \times 92.917\times10^6 \times 13.3 \times 10^3}{3.75^2 \times 10^6} = 867\times 10^3\,\text{N} \qquad n_{cr} = N_{cr}/N = 867/145.8 = 6$$

The maximum stress follows from: $\sigma_d = \dfrac{N_d}{A} \pm \dfrac{M_d \times z}{I} \times \dfrac{n_{cr}}{n_{cr} - 1}$.

$$\sigma_d = \frac{-145.8 \times 10^3}{50 \times 10^3} \pm \frac{16.5 \times 10^6}{92.917 \times 10^6} \times 55 \times \frac{6}{6 - 1} = -2.9 \pm 11.7\ \text{MPa}$$

Due to the strengthening the bending moments are decreased, but due to the post-tensioning the bending moments increase substantially. Post-tensioning the ties is needed if the slender ties are subjected to compressive normal forces. To increase the efficiency of the load transfer it is better to position the ties in such a way that the ties are tensioned continuously and post-tensioning the ties is not necessary. The following paragraph shows the design of a trussed vault.

§ 9.9 Trussed vault, following a circle segment

A circular vault can be strengthened with ties and two small radial struts as described before in chapter 6 for a parabolic vault. Structurally the trussed vault resembles the Polonceau truss.
For the design and to show the distribution of the loads an analyse is made for several loads. The parameters of the trussed arch are the span l = 2 a, the rise f and the angle γ for the web bars. The nodes are numbered clockwise. To simplify the analysis of the forces acting on the ties and struts the structure is assumed to be facetted and to be loaded only by concentrated forces acting on the joints.

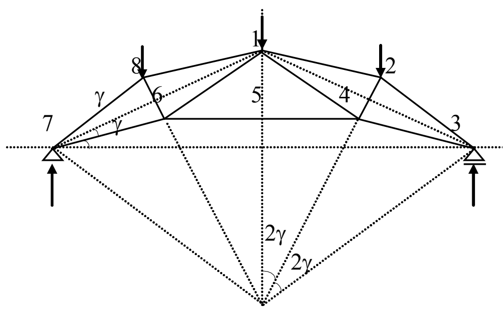

FIGURE 9.24 Trussed vault, geometry

Assume the angle between the ties and the horizontal line between the supports is equal to γ, the angle between the tie $S_{1\text{-}5}$ and $S_{6\text{-}5}$ is equal to 3γ, the angle between the chord $S_{7\text{-}8}$ and the tie $S_{6\text{-}7}$ is equal to 2γ.

The length of the span is equal to:	$l = 2\,a = 8\,R \sin\gamma \cos\gamma\,(2\cos^2\gamma - 1)$
The leverarm of the truss is equal to:	$f' = S_{1\text{-}5} = 2\,R \sin\gamma \sin(3\gamma) = 2\,R \sin^2\gamma\,(4\cos^2\gamma - 1)$
The length of the chords is equal to:	$S_{1\text{-}2} = S_{2\text{-}3} = S_{1\text{-}8} = S_{7\text{-}8} = 2\,R \sin\gamma$
The length of the struts is equal to:	$S_{2\text{-}4} = S_{6\text{-}8} = 4\,R \sin^2\gamma$

The structure is statically determinate. For every node the forces acting at the bars are defined with the vertical and horizontal equilibrium of the forces $\Sigma V = 0$ and $\Sigma H = 0$.

The procedure to define the forces acting at the bars is as following, define:
- the reaction forces with the vertical equilibrium of te truss, $\Sigma V = 0$.
- the thrust H with the bending moment around the top, $\Sigma M = 0$.
- successively the forces acting in the chords and ties with the equilibrium of the vertical and horizontal forces acting at the nodes, $\Sigma V = 0$ and $\Sigma H = 0$.

Concentrated load acting at the top

The truss is subjected to a concentrated load acting at the top: $F_1 = F$. Due to this load both vertical reaction forces are equal to $V = ½ F$. The thrust H fllows from the equilibrium of the bending moments around the top, $\Sigma M = 0$.

$$S_{456} = H f' = V a \quad \rightarrow \quad S_{456} = H = \frac{½ F \times 4 \times R \sin\gamma \cos\gamma\,(2\cos^2\gamma - 1)}{2 R \sin^2\gamma\,(4\cos^2\gamma - 1)}$$

$$S_{456} = H = \frac{F \cos\gamma\,(2\cos^2\gamma - 1)}{\sin\gamma\,(4\cos^2\gamma - 1)}$$

Successively for the nodes 1, 3 and 2 the normal forces acting at the bars are defined with the vertical and horizontal equilibrium of the forces $\Sigma V = 0$ and $\Sigma H = 0$.

Node 1: $\Sigma V_1 = 0$: $\quad S_{12} \sin\gamma - S_{14} \sin(3\gamma) = ½ F \quad \rightarrow \quad S_{14} = \frac{S_{12} \sin\gamma - ½ F}{\sin(3\gamma)}$

Node 1: $\Sigma H_1 = 0$: $\quad S_{12} \cos\gamma - S_{14} \cos(3\gamma) = H$

Substituting S_{14} and H: $\quad S_{12}\left[\cos\gamma - \frac{\sin\gamma \cos 3\gamma}{\sin(3\gamma)}\right] = \frac{-½ F}{\sin(3\gamma)} + \frac{F \cos\gamma\,(2\cos^2\gamma - 1)}{\sin\gamma\,(4\cos^2\gamma - 1)}$

Substituting $\sin(3\gamma) = \sin\gamma\,(4\cos^2\gamma - 1)$ and $\cos(3\gamma) = \cos\gamma\,(1 - 4\sin^2\gamma)$:

$$S_{12} = ¼ F / \sin\gamma$$

Next S_{12} is substituted into the expression for S_{14}: $\quad S_{14} = -\frac{¼ F}{\sin\gamma\,(4\cos^2\gamma - 1)}$.

Notice the sign is negative, so this element is compressed.

Node 3: $\Sigma H_3 = 0$: $\quad S_{23} \cos(3\gamma) = S_{34} \cos\gamma \quad \rightarrow \quad S_{34} = \frac{S_{23} \cos\gamma}{\cos(3\gamma)}$

Node 3: $\Sigma V_3 = 0$: $\quad S_{23} \sin(3\gamma) = ½ F + S_{34} \sin\gamma$

Substituting S_{34}: $\quad S_{23} = \frac{½F}{\sin(3\gamma)} + \frac{S_{23} \cos\gamma \sin\gamma}{\cos(3\gamma) \sin(3\gamma)}$

Substituting $\sin(3\gamma) = \sin\gamma\,(4\cos^2\gamma - 1)$ and $\cos(3\gamma) = \cos\gamma\,(1 - 4\sin^2\gamma)$:

$$S_{23} = \frac{F}{4\sin\gamma}$$

Next S_{23} is substituted into the expression for S_{34}: $\quad S_{34} = \frac{F\,(1 - 4\sin^2\gamma)}{4\sin\gamma}$

Node 2: $\Sigma V_2 = 0$: $S_{12} \sin\gamma + S_{24} \cos(2\gamma) = S_{23} \sin(3\gamma) \quad S_{24} = \frac{+S_{23} \sin(3\gamma) - S_{12} \sin\gamma}{\cos(2\gamma)}$

Next S_{23} and S_{12} are substituted into the expression for S_{24}: $\quad S_{24} = +½ F$ (tensioned)

Two concentrated loads acting at node 8 and 2

The truss is subjected to two concentrated loads acting at node 8 and 2, respectively F_8 and F_2. Due to these loads both vertical reaction forces are equal to $V = F$. The thrust H with the bending moment around the top, $\Sigma M = 0$.

$$S_{456} = H f' = F\,(a - 2 R \sin\gamma \cos\gamma) \quad \rightarrow \quad S_{456} = H = \frac{F \cos\gamma\,(1 - 4\sin^2\gamma)}{\sin^2\gamma\,(4\cos^2\gamma - 1)}$$

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Successively the normal forces acting at the bars are defined for the nodes 1, 3 and 2 with the vertical and horizontal equilibrium of the forces $\Sigma V = 0$ and $\Sigma H = 0$.

Node 1: $\Sigma V_1 = 0$: $S_{12} \sin\gamma - S_{14} \sin(3\gamma) = 0 \rightarrow S_{14} = \dfrac{S_{12} \sin\gamma}{\sin(3\gamma)}$

Node 1: $\Sigma H_1 = 0$: $S_{12} \cos\gamma - S_{14} \cos(3\gamma) = H$

Substituting S_{14} and H: $S_{12} \left(\cos\gamma - \dfrac{\sin\gamma \cos(3\gamma)}{\sin(3\gamma)}\right) = \dfrac{F \cos\gamma (1 - 4\sin^2\gamma)}{\sin\gamma (4\cos^2\gamma - 1)}$

Substituting $\sin(3\gamma) = \sin\gamma (4\cos^2\gamma - 1)$ and $\cos(3\gamma) = \cos\gamma (1 - 4\sin^2\gamma)$:

$$S_{12} = \frac{F (1 - 4\sin^2\gamma)}{2\sin\gamma}$$

Next S_{12} is substituted into the expression for S_{14}: $S_{14} = \dfrac{F (1 - 4\sin^2\gamma)}{2\sin\gamma (4\cos^2\gamma - 1)}$

Node 3: $\Sigma H_3 = 0$: $S_{23} \cos 3\gamma = S_{34} \cos\gamma \rightarrow S_{34} = \dfrac{S_{23} \cos\gamma}{\cos(3\gamma)}$

Node 3: $\Sigma V_3 = 0$: $S_{23} \sin(3\gamma) - S_{34} \sin\gamma = F$

Substituting S_{34}: $S_{23} = \dfrac{F}{\sin(3\gamma)} + \dfrac{S_{23} \cos\gamma \sin\gamma}{\cos 3\gamma \sin(3\gamma)}$

Substituting $\sin(3\gamma) = \sin\gamma (4\cos^2\gamma - 1)$ and $\cos(3\gamma) = \cos\gamma (1 - 4\sin^2\gamma)$:

$$S_{23} = \frac{F}{2\sin\gamma}$$

Next S_{23} is substituted into the expression for S_{34}: $S_{34} = \dfrac{F (1 - 4\sin^2\gamma)}{2\sin\gamma}$

Node 2: $\Sigma H_2 = 0$: $S_{12} \cos\gamma + S_{24} \sin(2\gamma) = S_{23} \cos(3\gamma)$

$$S_{24} = \frac{S_{23} \cos(3\gamma) - S_{12} \cos\gamma}{\sin(2\gamma)}$$

Next S_{23} and S_{12} are substituted into the expression for S_{24}: $S_{24} = 0$

Trussed vault, anti-metrical loaded.

The truss is subjected to an anti-metric loading. Due to this load the truss is subjected to respectively a load F acting upward at node 8 and a load F^- acting downward at node 2.
The reaction force acting at node 3 follows from the bending moment round node 7, $\Sigma M = 0$.

$$V_3 \times 2a = F (a + 2R \sin\gamma \cos\gamma) - F \times 2R \sin\gamma \cos(3\gamma) \rightarrow$$

$$V_3 = \frac{F \times \{4\sin\gamma \cos\gamma (2\cos^2\gamma - 1) + 2\sin\gamma \cos\gamma - 2\sin\gamma \cos\gamma (1 - 4\sin^2\gamma)\}}{8\sin\gamma \cos\gamma (2\cos^2\gamma - 1)}$$

$$V_3 = \frac{½ F}{(2\cos^2\gamma - 1)}$$

The thrust follows from the equilibrium of bending moments around the top, $\Sigma M = 0$:

$$S_{456} \times f' = \frac{F \times 4R \sin\gamma \cos\gamma (2\cos^2\gamma - 1)}{2 (2\cos^2\gamma - 1)} - F \times 2R \sin\gamma \cos\gamma \rightarrow S_{456} = 0$$

Successively the normal forces acting at the bars are defined for the nodes 1, 3 and 2 with the vertical and horizontal equilibrium of the forces ΣV = 0 and ΣH = 0.

Node 3: $\Sigma H_3 = 0$: $S_{23} \cos(3\gamma) = S_{34} \cos\gamma \rightarrow S_{34} = \frac{S_{23}\cos(3\gamma)}{\cos\gamma}$

Node 3: $\Sigma V_3 = 0$: $S_{23} \sin(3\gamma) - S_{34} \sin\gamma = \frac{F}{(2\cos^2\gamma - 1)}$

Substituting S_{34}: $S_{23} = \frac{F}{(2\cos^2\gamma - 1)\sin(3\gamma)} + \frac{S_{23}\cos(3\gamma)\sin\gamma}{\cos\gamma\sin(3\gamma)}$

Substituting $\sin(3\gamma) = \sin\gamma(4\cos^2\gamma - 1)$ and $\cos(3\gamma) = \cos\gamma(1 - 4\sin^2\gamma)$:

$$S_{23} = \frac{F}{4\sin\gamma(2\cos^2\gamma - 1)}$$

Next S_{23} is substituted into the expression for S_{34}: $S_{34} = \frac{F(1 - 4\sin^2\gamma)}{4\sin\gamma(2\cos^2\gamma - 1)}$

Node 1: $\Sigma H_1 = 0$: $S_{12}\cos\gamma - S_{14}\cos(3\gamma) = H = 0 \rightarrow S_{14} = \frac{S_{12}\cos\gamma}{\cos(3\gamma)}$

At the top of the truss a shear force is transferred, this force is equal to the load minus the reaction force (F – V):

$$(F - V) = \frac{F(4\cos^2\gamma - 3)}{2(2\cos^2\gamma - 1)}$$

Node 1: $\Sigma V_1 = 0$: $S_{12}\sin\gamma - S_{14}\sin(3\gamma) = -\frac{F(4\cos^2\gamma - 3)}{2(2\cos^2\gamma - 1)}$

Substituting S_{14}: $S_{12}\left[\sin\gamma - \frac{\cos\gamma\sin(3\gamma)}{\cos 3\gamma}\right] = -\frac{F\cos\gamma(4\cos^2\gamma - 3)}{2(2\cos^2\gamma - 1)}$

Substituting $\sin(3\gamma) = \sin\gamma(4\cos^2\gamma - 1)$ and $\cos(3\gamma) = \cos\gamma(1 - 4\sin^2\gamma)$:

$$S_{12} = \frac{F(4\cos^2\gamma - 3)(1 - 4\sin^2\gamma)}{4\sin\gamma(2\cos^2\gamma - 1)}$$

Next S_{12} is substituted into the expression for S_{14}: $S_{14} = \frac{F(4\cos^2\gamma - 3)}{4\sin\gamma(2\cos^2\gamma - 1)}$

Node 4: $\Sigma H_2 = 0$: $S_{14}\cos(3\gamma) + S_{24}\sin(2\gamma) = S_{34}\cos\gamma \rightarrow S_{24} = \frac{S_{34}\cos\gamma - S_{14}\cos(3\gamma)}{\sin(2\gamma)}$

Next S_{23} and S_{12} are substituted into the expression for S_{24}: $S_{24} = \frac{F(4\cos^2\gamma - 3)}{2 \times (2\cos^2\gamma - 1)}$

The following table summarizes the resulting forces for the trussed vault described with the parameter γ. For low rise vaults cos γ is approximately equal to 1, then a reduction of the ratio f/a will increase the forces acting on the elements nearly linearly.

member	F_1	F_8 and F_2	Anti-metrisch F_8 and F_2^-
H =	$+\frac{F\cos\gamma\ (2\cos^2\gamma - 1)}{\sin\gamma\ (4\cos^2\gamma - 1)}$	$+\frac{F\cos\gamma\ (4\cos^2\gamma - 1)}{\sin\gamma\ (4\cos^2\gamma - 1)}$	0
S_{12} =	$-\frac{\frac{1}{4}F}{\sin\gamma}$	$-\frac{F\,(1 - 4\sin^2\gamma)}{2\sin\gamma}$	$-\frac{F(4\cos^2\gamma - 3)\times(1 - 4\sin^2\gamma)}{4\sin\gamma\ (2\cos^2\gamma - 1)}$
S_{23} =	$-\frac{\frac{1}{4}F}{\sin\gamma}$	$-\frac{\frac{1}{2}F}{\sin\gamma}$	$-\frac{F}{4\sin\gamma\,(2\cos^2\gamma - 1)}$
S_{14} =	$-\frac{\frac{1}{4}F}{\sin\gamma\ (4\cos^2\gamma - 1)}$	$+\frac{\frac{1}{2}F\,(1 - 4\sin^2\gamma)}{\sin\gamma\,(4\cos^2\gamma - 1)}$	$+\frac{F\ 4\cos^2\gamma - 3)}{4\sin\gamma\,(2\cos^2\gamma - 1)}$
S_{34} =	$+\frac{F\,(1 - 4\sin^2\gamma)}{4\sin\gamma}$	$+\frac{F\,(1 - 4\sin^2\gamma)}{2\sin\gamma}$	$+\frac{F\ (1 - 4\sin^2\gamma)}{4\sin\gamma\,(2\cos^2\gamma - 1)}$
S_{456} =	$+\frac{F\cos\gamma\,(2\cos^2\gamma - 1)}{\sin\gamma\,(4\cos^2\gamma - 1)}$	$+\frac{F\cos\gamma\ (4\cos^2\gamma - 1)}{\sin\gamma\,(4\cos^2\gamma - 1)}$	0
S_{24} =	$+\frac{1}{2}F$	0	$-\frac{F\,(4\cos^2\gamma - 3)}{2\ (2\cos^2\gamma - 1)}$
S_{18} =	$-\frac{\frac{1}{4}F}{\sin\gamma}$	$-\frac{F\,(1 - 4\sin^2\gamma)}{2\sin\gamma}$	$+\frac{F\,(4\cos^2\gamma - 3)\,(1 - 4\sin^2\gamma)}{4\sin\gamma\ (2\cos^2\gamma - 1)}$
S_{78} =	$-\frac{\frac{1}{4}F}{\sin\gamma}$	$-\frac{\frac{1}{2}F}{\sin\gamma}$	$-\frac{F}{4\sin\gamma\,(2\cos^2\gamma - 1)}$
S_{16} =	$-\frac{\frac{1}{4}F}{\sin\gamma\,(4\cos^2\gamma - 1)}$	$+\frac{\frac{1}{2}F\,(1 - 4\sin^2\gamma)}{\sin\gamma\,(4\cos^2\gamma - 1)}$	$-\frac{F\,(4\cos^2\gamma - 3)}{4\sin\gamma\ (2\cos^2\gamma - 1)}$
S_{67} =	$+\frac{F.(1 - 4\sin^2\gamma)}{4\sin\gamma}$	$+\frac{F\,(1 - 4.\sin^2\gamma)}{2\sin\gamma}$	$-\frac{F\,(1 - 4\sin^2\gamma)}{4\sin\gamma\,(2\cos^2\gamma - 1)}$
S_{68} =	$+\frac{1}{2}F$	0	$+\frac{F\,(4\cos^2\gamma - 3)}{2\,(2\cos^2\gamma - 1)}$

TABLE 9.10 Forces acting in the trussed vault subjected to concentrated loads F_i acting at the nodes.

The following table summarizes the resulting forces for the trussed vault with tan $2\gamma = 0.25$ and $\gamma = 7.018^{o}$. For this angle γ the cos γ is nearly equal to 1: $\cos\ \gamma = 0.9925 \approx 1$. For this low rise trussed vault a reduction of the ratio f/a will increase the forces acting on the elements nearly linearly.

Due to the concentrated load acting at the top the ties S_{14} and S_{18} are subjected to compression: $S_{14} = S_{16} = -0.696$ and due to the permanent load these ties are tensioned: $S_{14} = S_{16} = +0.613$. The tie is subjected to a tensile normal force if the live load is smaller than 88% of the permanent load.
Due to the asymmetrical load acting at node 1 and 2 the tie S_{16} is subjected to compression $S_{16} = -0.685$ and due to the permanent load this tie is tensioned $S_{16} = +0.613$. The tie is tensioned if the live load is smaller than 90% of the live load.
The trussed vault can be strengthened with ties if for the ultimate state the permanent load is substantial larger than the live load. Assuming the load factor for the permanent load is 0.9 and the load factor for the live load is 1.5 then the representative permanent load has to be at least a factor $1.5/(0.9\times0.88) = 1.89$ larger than the representative live load.

Member	F_1	F_8 and F_2	F_8, F_1 and F_2	Anti-metrisch F_8 and F_2	Asymmetrisch F_2	Asymmetrisch ½ F_1 + F_2
H =	+2.721	+2.598	+5.319	0	+1.299	+2.659
S_{12} =	-2.046	-3.848	-5.894	-1.865	-2.856	-3.879
S_{23} =	-2.046	-4.092	-6.138	-2.109	3.101	-4.123
S_{14} =	-0.696	+1.309	+0.613	+1.983	+1.646	+1.298
S_{34} =	+1.924	+3.848	+5.772	+1.983	2.916	+3.877
S_{456} =	+2.721	+2.598	+5.319	0	+1.299	+2.659
S_{24} =	+0.500	0	+0.500	-0.485	-0.242	+0.007
S_{18} =	-2.046	-3.848	-5.894	+1.865	-0.992	-2.015
S_{78} =	-2.046	-4.092	-6.138	+2.109	-0.992	-2.015
S_{16} =	-0.696	+1.309	+0.613	-1.983	-0.337	-0.685
S_{67} =	+1.924	+3.848	+5.772	-1.983	+0.932	+1.894
S_{68} =	+0.500	0	+0.500	+0.485	+0.242	+0.492

TABLE 9.11 Trussed vault with tan 2γ = 0,25 and γ = 7.018, subjected to concentrated loads acting at the nodes .

§ 9.10 Radial ties

To design a Form-Active-Structure optimal the form of the structure has to follow the line of the thrust. For a structure subjected to several varying loads the lines of the thrust will vary. To reduce the bending moments the form of the structure has to approach the varying lines as much as possible. For a circular vault the line of the system is identical to the line of the thrust if this vault is subjected to an equally distributed radial load, for any other loading the structure will be subjected to bending. Consequently the load transfer is not optimal for a circular vault, subjected to varying loads.

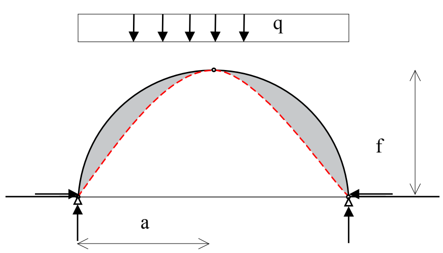

FIGURE 9.25 Line of thrust and bending moments for a circular vault subjected to equally distributed load

Any load can be solved into a radial and tangential component. An equally distributed surface load can be solved into an equally distributed radial load and a tangential load. At the top the vertical surface load is acting radial. At another point of the surface between the top and the supports the vertical load is not acting radially and can be solved into a radial component and a component acting tangentially.

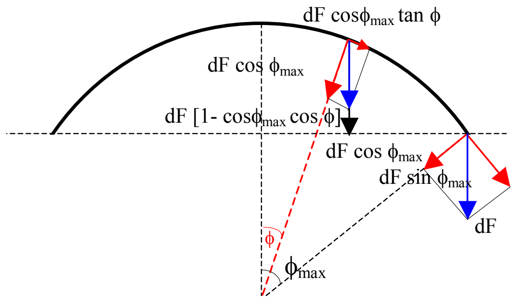

FIGURE 9.26 Solving a vertical surface load into a radial, vertical and horizontal load

To define these loads a small part of the surface R dϕ is considered. This part is subjected to a vertical force dF = qR dϕ. This vertical force is solved into a radial component dF cos ϕ and a tangential component dF sin ϕ. The radial component decreases from the crown to the support. Thus for any part of the surface the vertical load dF can be solved into a constant equally distributed radial component and a tangential component. The constant equally distributed radial load is equal to the minimal radial load, following from the radial load defined for $\phi = \phi_{max}$. Firstly the constant radial load is defined. At the support, for $\phi = \phi_{max}$, the vault is subjected to a vertical force dF = q R dϕ. This vertical force is solved into a radial component dF cos ϕ_{max} and a tangential component dF sin ϕ_{max}. Next this tangential component is solved in a horizontal component dF sin ϕ_{max} cos ϕ_{max} and a vertical component dF $\sin^2 \phi_{max}$.
For a small piece of the surface R dϕ, with $\phi < \phi_{max}$, a part of the vertical load dF (equal to dF cos ϕ_{max}/cos ϕ) is solved into a constant radial load equal to dFcos ϕ_{max} and a tangential load equal to dF cos ϕ_{max} tan ϕ.
Next this tangential load is solved into a horizontal and a vertical component respectively

$$dF \cos \phi_{max} \sin \phi \text{ and } dF \cos \phi_{max} \sin^2\phi / \cos \phi.$$

The remaining vertical component is equal to:

$$dF_v = dF\,[1 - \cos \phi_{max} / \cos \phi] \quad [9.58]$$

Dividing the forces by R dϕ gives the vertical, horizontal and radial loads acting at the surface:

$$q_r = q \cos \phi_{max} \quad [9.59]$$

$$q_h = q \cos \phi_{max} \sin \phi \quad [9.60]$$

$$q_v = q\,[1 - \cos \phi_{max}/\cos \phi + \cos \phi_{max} \sin^2\phi/\cos \phi] = q\,[1 - \cos \phi_{max} \cos \phi] \quad [9.61]$$

Due to the radial load the vault is subjected to a normal force N = q_r and due to the vertical and horizontal components the vault is subjected to a normal force and bending.

Strengthening circular vaults with ties running from a centre point

Strengthening a circular vault with ties is efficient if the load is solved into a radial component and components parallel to the ties. The following design shows a vault strengthened with slender ties running from a centre point, above the supports at a distance Δ, to tension the ties permanently. The position of the centre above the supports reduces the lever arm, between the top and the centre and increases the thrust. To design this structure optimal the distance Δ has to be chosen carefully, to tension the ties continuously for all load combinations.

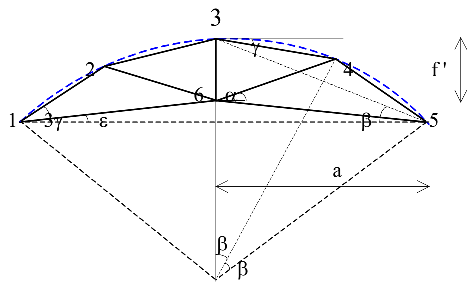

FIGURE 9.27 Vault strengthened with radial ties.

Next the effect of the position of the centre is analysed. To define the transfer of the loads analytically the structure is simplified and schemed as a facetted truss with hinged members. As before the span is equal to 2 a with $a = 2 R \sin \beta \cos \beta$. The rise of the vault is equal to f, with $f = 2 R \sin^2 \beta$. The length of the chords is equal to $2 R \sin \gamma$ with $\gamma = ½ \beta$. The length of the vertical tie halfway the span is equal to $f' = f - \Delta$, with $\Delta = a \tan \varepsilon$. The structure is subjected to concentrated forces F acting at the nodes. The normal forces acting in the ties and chords are defined with the equilibrium of the bending moments and forces acting horizontal and vertical conform the global positioning system of the coordinates. The centre of the coordinates is positioned at the top. Compressed elements are negative and tensioned elements are positive.

Vault subjected to a concentrated vertical force acting at the top.

The vault is loaded at the top by a vertical force F. The reaction forces are: $V_1 = V_5 = ½ F$

node 4, the thrust acting at node 4 follows from: $H = + ½ F \times a/f'$

Node 3, $\Sigma H = 0$: $S_{12} \cos \gamma = - H$ → $S_{12} = -H/\cos \gamma$, compression

Node 3, $\Sigma V = 0$: $S_{36} = - F - 2 S_{12} \sin \gamma$

Substituting $S_{12} = -H/\cos \gamma$ → $S_{36} = - F + 2 H \tan \gamma$

Node 6, $\Sigma V = 0$: $S_{46} \sin \alpha + ½ S_{36} - S_{56} \sin \varepsilon = 0$ → $S_{46} = \dfrac{S_{56} \sin \varepsilon - ½ S_{36}}{\sin \alpha}$

Node 6, $\Sigma H = 0$: $+ S_{56} \cos \varepsilon + S_{46}.\cos \alpha = H$

Substituting S_{46}: $S_{56} \cos \varepsilon = H - \dfrac{S_{56} \sin \varepsilon - ½ S_{36}}{\tan \alpha}$ $S_{56} = \dfrac{H + ½ S_{14}/\tan \alpha}{\sin \varepsilon/\tan \alpha + \cos \varepsilon}$

$S_{46} = \dfrac{H - S_{56} \cos \varepsilon}{\cos \alpha}$

Node 5, $\Sigma H = 0$: $S_{45} \cos (3 \gamma) = -S_{56} \cos \varepsilon$ → $S_{45} = -S_{34} \cos \varepsilon/\cos (3 \gamma)$ compression

Vault subjected to two Loads acting at node 2 and 4.

The vault is subjected to concentrated loads acting at node 2 and node 6.
The reaction forces are: $V_1 = V_5 = F$
node 4: the thrust acting at node 4 follows from: $H = + (F a - F R \sin \beta)/f'$

node 3, $\Sigma H = 0$: $S_{34} \cos \gamma = -H$ → $S_{34} = -H/\cos \gamma$ (compression) node 3, $\Sigma V = 0$: $S_{36} - 2 S_{34} \sin\gamma = 0$ $\rightarrow$ $\frac{1}{2} S_{36} = + H \tan\gamma$ (tension)

Node 6, $\Sigma V = 0$: $S_{46} \sin a + \frac{1}{2} S_{36} - S_{56} \sin\varepsilon = 0$ $\rightarrow$ $S_{46} = \dfrac{S_{56} \sin\varepsilon - \frac{1}{2} S_{36}}{\sin\alpha}$

Node 6, $\Sigma H = 0$: $+ S_{56} \cos\varepsilon + S_{46} \cos\alpha = H$

Substituting S_{46}: $S_{56} \cos\varepsilon = H - \dfrac{S_{56} \sin\varepsilon - \frac{1}{2} S_{36}}{\tan\alpha}$ $\rightarrow$ $S_{56} = \dfrac{H + \frac{1}{2} S_{14}/\tan\alpha}{\sin\varepsilon/\tan\alpha + \cos\varepsilon}$

$S_{46} = \dfrac{H - S_{56} \cos\varepsilon}{\cos\alpha}$

Node 5, $\Sigma H = 0$, $S_{45} \cos(3\gamma) = -S_{56} \cos\varepsilon$ $\rightarrow$ $S_{45} = -S_{56} \cos\varepsilon/\cos(3\gamma)$

Asymmetrical load.

To define the forces acting on the elements due to an asymmetrical load, the analysis is simplified by splitting up this load case into an symmetrical and an anti-metrical load. Firstly the vault is loaded anti-metrically, at node 2 by a concentrated vertical force F acting downward and at node 6 by a vertical force F acting upward. The results of an asymmetrical load are found by dividing the sum of the results of the anti-metrical and the symmetrical loads by 2.

Vault subjected to two Loads acting at node 2 and 4 anti-metrically.

The reaction force V_5 follows from:

$$V_5 = \frac{F(a + R\sin\beta) - F(a - R\sin\beta)}{2R \times 2\sin\beta\cos\beta} = \frac{1}{2} F/\cos\beta, \quad V_1 = -\frac{1}{2} F/\cos\beta.$$

The thrust acting at node 6 follows from: $H = \dfrac{F R \sin\beta - 2 F R \sin\beta \cos\beta/(2\cos\beta)}{f'} = 0$

Node 3, $\Sigma H = 0$: $S_{34} \cos\gamma = H$, Substituting $H = 0$ $\rightarrow$ $S_{34} = 0$

Node 3, $\Sigma V = 0$: $S_{34} \sin\gamma + \frac{1}{2} S_{36} = 0$, $\rightarrow$ $S_{36} = -2 S_{34} \sin\gamma = 0$

Node 4, $\Sigma H = 0$: $S_{46} \cos\alpha + S_{45} \cos(3\gamma) = 0$ $\rightarrow$ $S_{46} = -|S_{45}| \cos(3\gamma)/\cos\alpha$ (compressive)

Node 4, $\Sigma V = 0$: $+ -S_{46} \sin\alpha + S_{45} \sin(3\gamma) = F$

Substituting $|S_{46}|$: $S_{45} = \dfrac{-F}{\sin(3\gamma) + \cos(3\gamma)\tan\alpha} < 0$ (compressive)

Node 5, $\Sigma H = 0$: $S_{45} \cos(3\gamma) + S_{56} \cos\varepsilon = 0$ $\rightarrow$ $S_{56} = +\dfrac{|S_{45}| \cos(3\gamma)}{\cos\varepsilon} > 0$ (tension)

node 3, $\Sigma H = 0$: $S_{23} \cos\gamma = H$, Substituting $H = 0$ gives $S_{23} = 0$

In the same way the forces acting at node 5 are defined: $S_{26} = + |S_{46}|$; $S_{12} = +|S_{45}|$; $S_{16} = -|S_{56}|$

To define the forces for an asymmetrical load the results of the anti-metrical loads and symmetrical load are added and divided by 2. The following table shows the results of the analyse.

	Symmetrical forces	Symmetrical force	Anti-metrical
	F_2, F_4	F_1	$F_2, - F_6$
V_1	$-F$	$-\frac{1}{2}F$	$+\frac{1}{2}F/\cos\beta$
V_5	$-F$	$-\frac{1}{2}F$	$-\frac{1}{2}F/\cos\beta$
H	$+F(a - R\sin\beta)/f'$	$+\frac{1}{2}F\,a/f'$	0
S_{34}	$-H/\cos\gamma$	$-\frac{1}{2}H/\cos\gamma$	0
S_{36}	$+2H\tan\gamma$	$-F + 2H\tan\gamma$	0
S_{23}	$-H/\cos\gamma$	$-\frac{1}{2}H/\cos\gamma$	0
S_{56}	$+\frac{H\tan\alpha + \frac{1}{2}S_{36}}{\sin\varepsilon + \cos\varepsilon\tan\alpha}$	$+\frac{H\tan\alpha + \frac{1}{2}S_{36}}{\sin\varepsilon + \cos\varepsilon\tan\alpha}$	$+\frac{\lvert S_{45}\rvert\cos(3\gamma)}{\cos\varepsilon}$
S_{45}	$-\frac{S_{56}\cos\varepsilon}{\cos(3\gamma)}$	$-\frac{S_{56}\cos\varepsilon}{\cos(3\gamma)}$	$-\frac{F}{\sin(3\varepsilon) + \cos(3\gamma)\tan\alpha}$.
S_{46}	$+\frac{H - S_{56}\cos\varepsilon}{\cos\alpha}$	$+\frac{H - S_{56}\cos\varepsilon}{\cos\alpha}$	$-\frac{\lvert S_{45}\rvert\cos\varepsilon}{\cos\alpha}$
S_{16}	$\frac{H\tan\alpha + \frac{1}{2}S_{56}}{\sin\varepsilon + \cos\varepsilon\tan\alpha}$	$\frac{H\tan\alpha + \frac{1}{2}S_{56}}{\sin\varepsilon + \cos\varepsilon\tan\alpha}$	$-\frac{\lvert S_{45}\rvert\cos(3\gamma)}{\cos\varepsilon}$
S_{26}	$+\frac{H - S_{56}\cos\varepsilon}{\cos\alpha}$	$\frac{H - S_{56}\cos\varepsilon}{\cos\alpha}$	$+\frac{\lvert S_{56}\rvert\cos\varepsilon}{\cos\alpha}$
S_{12}	$-\frac{S_{56}\cos\varepsilon}{\cos(3\gamma)}$	$-\frac{S_{56}\cos\varepsilon}{\cos(3\gamma)}$	$+\frac{F}{\sin(3\varepsilon) + \cos(3\gamma)\tan\alpha}$

TABLE 9.12 Forces acting at the members of the radial reinforced trussed vault

Example vault strengthened with radial ties

The forces are defined for a vault with $f = a/4$, radius $R = 2.125 \times a$, $\tan\beta = f/a$, so $\beta = 14.036^0$ and $\gamma = 7.018^0$. The structure is subjected to concentrated forces acting at the nodes: F = 1.0 kN. For this vault the distance of the centre point above the horizontal line between the supports is equal to $\Delta = \frac{1}{4}$ f, thus the lever arm f' is equal to $0.75 \times f$, so $f' = {}^{3}/_{16}\,a$. The following table shows the forces due to the symmetrical and asymmetrical loads acting at the nodes, for F = 1.0 kN

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Member	Sym.	Sym.	Permanent.	Anti-metrical	Asymmetrical.
	F_2, F_4	F_3	F_2, F_3, F_4	$\frac{1}{2}\times(-F_2, F_4)$	$\frac{1}{2}\times(F_2, F_4) + \frac{1}{2}\times(-F_2, F_4)$
S_{34}	- 2.604	- 2.687	- 5.291	0	- 1.302
S_{36}	+0.636	- 0.343	+0.293	0	+0.318
S_{46}	- 0.531	+1.148	+0.617	-0.822	- 1.088
S_{56}	+3.107	+1.554	+4.661	+0.801	+2.354
S_{45}	-3.323	- 1.662	+4.985	-0.856	- 2.518
S_{26}	-0.531	+1.148	+0.617	+0.822	- 0.556
S_{23}	- 2.604	- 2.687	- 5.291	0	- 1.302
S_{16}	+3.107	+1.554	+4.661	-0.801	+0.753
S_{12}	- 3.323	-1.662	+4.985	+0.856	- 0.805

TABLE 9.13 Forces acting in the members of the radial reinforced vault

A concentrated load acting at the crown causes a compressive force in the vertical web bar: S_{36} = -0.343, but the permanent load will cause a tensile force +F = 0.293. Thus the web bar will be tensioned if the concentrated load is not larger than 85% of the permanent load.

Due to the asymmetrical concentrated load acting at node 4 member S_{46} is compressed, $_{,}F$ = -1.088. The forces in this tie due to this asymmetrical load is compensated by the permanent load if the asymmetrical load is smaller than the permanent load.

For a vault with a span 2 a = 2 × 7.2 m and f = 1.8 m the loads acting at the nodes are:

Permanent load: $F_i = q_g R \beta = 3.2 \times 15.3 \times (14.036) \times \pi/180 = 12$ kN

Live load: $F_3 = ½\, q_e R \sin\beta = ½ \times 5.0 \times 15.3 \times 0.47059 = 9.28$ kN

$F_4 = ½\, q_e R \sin\beta + (a - q_e R \sin\beta)/2 = 5.0 \times ½ \times 7.2 = 18.0$ kN

The normal forces acting at member S_{46} are:

permanent load: $S_{46} = 0.617 \times 12.0 = +7.4$ kN

asym. live load: $S_{46} = -1.088 \times 18.0 + 1.148 \times 9.28 = -8.93$ kN, S_{46} is compressed.

To avoid compressive normal force acting in this tie the dead load has to be increased. The analysis shows the effect of decreasing the lever arm to avoid compressive loads acting in the web bars. To prevent the web bars to be subjected by compressive normal forces the designer can decrease the lever arm or increase the dead load.

Computer calculation for the truss

To check the analysis the forces, acting at the members of the truss, are calculated with a Finite-Element program. Again concentrated forces acting at the nodes load the truss. Comparing the results shows that the results of the finite element calculation match with the analyse

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Node	x =	y =	Member	Top: F_3	F_2, F_4	F_4	F_2, F_3, F_4
n1	-7.20	1.800	(S_{12}) : n1-n2	-1.66	-3.32	- 0.81	-4.98
n2	-3.71	0.457	(S_{23}) : n2-n3	-2.69	-2.60	- 1.30	-5.29
n3	0	0	(S_{34}) : n3-n4	-2.69	-2.60	- 1.30	-5.29
n4	3.71	0.457	(S_{45}) : n4-n5	-1.66	-3.32	-2.52	-4.98
n5	7.20	1.800	(S_{56}) : n5-n6	+1.55	+3.11	+2.36	+4.66
n6	0	1.35	(S_{16}) : n6-n1	+1.55	+3.11	+0.75	+4.66
			(S_{26}) : n2-n6	+1.15	-0.53	+0.56	+0.62
			(S_{36}) ; n3-n6	-0.34	+0.64	+0.32	+0.30
			(S_{46}) : n4-n6	+1.15	-0.53	- 1.09	+0.62

TABLE 9.14 Coordinates and forces acting at the members of the truss

Simplification: $\varepsilon = \gamma$

The diagonals are straight if the angle ε, between the lower ties and the horizontal line through the supports, is equal to $\varepsilon = \gamma = ½\,\beta$. For this angle the distance Δ follows from $\Delta = a \tan\gamma$. Choosing the angle equal to $\gamma = ½\,\beta$ simplifies the analysis of the forces. To define the transfer of the loads the structure is simplified and schemed as a facetted truss with hinged members. The structure is subjected to concentrated forces F acting at the nodes. The normal forces acting in the ties and chords

are defined with the equilibrium of the bending moments and forces acting horizontal and vertical conform the global positioning system of the coordinates. The centre of the coordinates is positioned at the top. Compressed elements are negative and tensioned elements are positive.
.As before the span is equal to 2 a with $a = 2 R \sin \beta \cos \beta$. The rise of the vault is equal to f, with $f = 2 R \sin^2 \beta$. the length of the chords is equal to $2 R \sin \gamma$ with $\gamma = ½ \beta$. The length of the vertical tie halfway the span S_{36} is equal to the length of the lever arm f ', with: $f' = 2 \times (2 R \sin \beta) \times \tan \gamma = 4 R \sin^2\gamma$

Vault subjected to concentrated load acting at the top.

The vault is loaded at the top by a vertical force F. The reaction forces are: $V_1 = V_5 = ½ F$
The thrust acting at node 4 follows from: $H = + ½ F \times a / f'$

$$H = \frac{½ F \times (2 R \sin \beta \cos \beta)}{4 R \sin^2\gamma} \rightarrow H = + \frac{½ F \cos \beta \cos \gamma}{\sin \gamma}$$

Node 3, SH = 0: $S_{12} \cos \gamma = - H \rightarrow S_{12} = - \frac{½ F \cos \beta}{\sin \gamma}$ (compressive)

Node 3, ΣV = 0: $S_{36} = - F - 2 S_{12} \sin \gamma \rightarrow S_{36} = - F - F \cos \beta \rightarrow S_{36} = - F \sin^2\gamma$ (compressive)

Node 6, SH = 0: $+ S_{56} \cos \gamma + S_{46} \cos \gamma = H$ $\quad S_{46} = \frac{½ F \cos \beta \cos \gamma}{\sin \gamma} - S_{56}$.

Node 6, ΣV = 0: $S_{46} \sin \gamma + ½ S_{36} - S_{56} \sin \gamma = 0$
Substituting S_{46} and S_{36}: $\quad S_{56} = \frac{F (1 - 4 \sin^2\gamma)}{4 \sin \gamma} \rightarrow S_{46} = F/(4 \sin \gamma)$

Node 4, ΣV = 0: $S_{45} \sin (3 \gamma) = S_{34} \sin \gamma + S_{46} \sin \gamma \rightarrow$

$S_{45} = - \frac{F (3 - 4 \sin^2\gamma)}{\sin (3 \gamma)}$ (compression)

Vault subjected to two Loads acting at node 2 and 4.

The vault is subjected to concentrated loads acting at node 2 and node 6.
The reaction forces are: $V_1 = V_5 = F$

The thrust acting at node 6 follows from: $H = + (F \times a - F R \sin \beta) / f'$

$$H = \frac{½ F \times (2 R \sin \beta \cos \beta - R \sin \beta)}{4 R \sin^2\gamma} \rightarrow H = + \frac{½ F \cos \gamma (4 \cos^2\gamma - 3)}{\sin \gamma}$$

Node 3, ΣH = 0: $S_{34} \cos \gamma = -H \rightarrow S_{34} = \frac{-½ F (4 \cos^2\gamma - 3)}{\sin \gamma}$ (compression)

Node 3, ΣV = 0: $S_{36} - 2 S_{34} \sin \gamma = 0 \rightarrow S_{36} = F (4 \cos^2\gamma - 3)$ (tension)

Node 6, ΣH = 0: $+ S_{56} \cos \varepsilon + S_{46} \cos \alpha = H \rightarrow S_{46} = \frac{F (4 \cos^2 \gamma - 3)}{2 \sin \gamma} - S_{56}$

Node 6, ΣV = 0: $S_{46} \sin \alpha + ½ S_{36} - S_{56} \sin \varepsilon = 0 \rightarrow S_{56} = \frac{½ F (4 \cos^2 \gamma - 3)}{\sin \gamma}$

Substituting S_{56} into the expression for S_{46} gives: $S_{46} = 0$

Node 4, $\Sigma V = 0$, $S_{45} \sin(3\gamma) = S_{34} \sin\gamma + F + S_{46} \sin\gamma \rightarrow S_{45} = -\frac{\frac{1}{2} F (4\cos^2\gamma - 1)}{\sin(3\gamma)}$

Asymmetrical load.

To define the forces acting on the elements due to an asymmetrical load, the analysis is simplified by splitting up this load case into an symmetrical and an anti-metrical load. Firstly the vault is loaded anti-metrically; at node 2 by a concentrated vertical force F acting upward and at node 4 by a vertical force F acting downward. The results of an asymmetrical load are found by adding the results of the anti-metrical loads and symmetrical loads and divided by 2.

Two Loads acting at node 2 and 4 anti-metrically.

The reaction force V_3 follows from: $V_5 = \frac{2 F R \sin\beta}{2 R\, 2 \sin\beta \cos\beta} = \frac{1}{2} F/\cos\beta$, $V_1 = -\frac{1}{2} F/\cos\beta$.

The thrust acting at node 6 follows from: $H = \frac{F R \sin\beta - F\, 2 R \sin\beta \cos\beta/(2\cos\beta)}{f''} = 0$

Node 3, $\Sigma H = 0$: $S_{34} \cos\gamma = H$, Substituting $H = 0 \rightarrow S_{34} = 0$

Node 3, $\Sigma V = 0$: $S_{34} \sin\gamma + \frac{1}{2} S_{36} = 0$, $\rightarrow S_{36} = 0$

Node 6: $\Sigma H = 0$: $S_{46} \cos\gamma + S_{56} \cos\gamma = 0$

Node 4, $\Sigma V = 0$: $S_{45} \sin(3\gamma) = F - S_{46} \sin\gamma \rightarrow S_{45} \sin(3\gamma) = F - S_{46} \sin\gamma$

Node 5, $\Sigma V = 0$: $S_{45} \sin(3\gamma) = \frac{1}{2} F/\cos\beta + S_{56} \sin\gamma \rightarrow S_{45} \sin(3\gamma) = \frac{1}{2} F/\cos\beta + S_{56} \sin\gamma$

Thus: $F - S_{46} \sin\gamma = \frac{1}{2} F/\cos\beta + S_{56} \sin\gamma$

Substituting; $S_{56} = -S_{46}$ $2 S_{46} \sin\gamma = -F + \frac{1}{2} F/\cos\beta$

$$S_{46} = -\frac{F(4\cos^2\gamma - 3)}{4\sin\gamma\,(2\cos^2\gamma - 1)}$$

$$S_{56} = +\frac{F(4\cos^2\gamma - 3)}{4\sin\gamma\,(2\cos^2\gamma - 1)}$$

In the same way the forces acting at node 5 are defined: $S_{26} = +|S_{46}|$; $S_{12} = +|S_{45}|$; $S_{16} = -|S_{56}|$

To define the forces for an asymmetrical load the results of the anti-metrical loads and symmetrical load are added and divided by 2.

Results

The following table shows for the trussed vault the results of the analysis.

	Symmetrical forces	Symmetrical force	Anti-metrical load	Asymmetrical load
	F_2, F_4	F_1	$-F_2, F_4$	F_4
V_1	$-F$	$-\frac{1}{2}F$	$+\frac{1}{2}F/\cos(2\gamma)$	$-{}^1/_4 F + {}^1/_4 F/\cos(2\gamma)$
V_5	$-F$	$-\frac{1}{2}F$	$-\frac{1}{2}F/\cos(2\gamma)$	$-{}^1/_4 F - {}^1/_4 F/\cos(2\gamma)$
H	$+\frac{F\cos\gamma\,(4\cos^2\gamma - 3)}{2\sin\gamma}$	$\frac{F\cos\gamma\,(2\cos^2\gamma - 1)}{2\sin\gamma}$	0	$\frac{F\cos\gamma\,(4\cos^2\gamma - 3)}{4\sin\gamma}$
S_{34}	$-\frac{F(4\cos^2\gamma - 3)}{2\sin\gamma}$	$-\frac{F(2\cos^2\gamma - 1)}{2\sin\gamma}$	0	$-\frac{F(4\cos^2\gamma - 3)}{4\sin\gamma}$
S_{36}	$-F(4\cos^2\gamma - 3)$	$-2F\sin^2\gamma$	0	$-\frac{1}{2}F(4\cos^2\gamma - 3)$
S_{23}	$-\frac{F(4\cos^2\gamma - 3)}{2\sin\gamma}$	$-\frac{F(2\cos^2\gamma - 1)}{2\sin\gamma}$	0	$+\frac{F(4\cos^2\gamma - 3)}{4\sin\gamma}$
S_{56}	$+\frac{F(4\cos^2\gamma - 3)}{2\sin\gamma}$	$+\frac{F(1 - 4\sin^2\gamma)}{4\sin\gamma}$	$+\frac{F(4\cos^2\gamma - 3)}{4(2\cos^2\gamma - 1)\sin\gamma}$	$+\frac{F(4\cos^2\gamma - 3)\times(4\cos^2\gamma - 1)}{8(2\cos^2\gamma - 1)\sin\gamma}$
S_{45}	$-\frac{F(4\cos^2\gamma - 1)}{2\sin(\gamma)}$	$-\frac{F(3 - 4\sin^2\gamma)}{4\sin(3\gamma)}$	$-\frac{F(4\cos^2\gamma - 1)}{4(2\cos^2\gamma - 1)\sin(3\gamma)}$	$-\frac{F(4\cos^2\gamma - 1)\times(4\cos^2\gamma - 1)}{8(2\cos^2\gamma - 1)\sin(3\gamma)}$
S_{46}	0	$\frac{F}{4\sin\gamma}$	$-\frac{F(4\cos^2\gamma - 3)}{4(2\cos^2\gamma - 1)\sin\gamma}$	$-\frac{F(4\cos^2\gamma - 3)}{8(2\cos^2\gamma - 1)\sin\gamma}$
S_{16}	$+\frac{F(4\cos^2\gamma - 3)}{2\sin\gamma}$	$+\frac{F(1 - 4\sin^2\gamma)}{4\sin\gamma}$	$\frac{F(4\cos^2\gamma - 3)}{4(2\cos^2\gamma - 1)\sin\gamma}$	$-\frac{F(4\cos^2\gamma - 3)\times(4\cos^2\gamma - 3)}{8(2\cos^2\gamma - 1)\sin\gamma}$
S_{26}	0	$\frac{F}{4\sin\gamma}$	$+\frac{F(4\cos^2\gamma - 3)}{4(2\cos^2\gamma - 1)\sin\gamma}$	$+\frac{F(4.\cos^2\gamma - 3)}{8(2\cos^2\gamma - 1)\sin\gamma}$
S_{12}	$-\frac{F(4\cos^2\gamma - 1)}{2\sin(3\gamma)}$	$-\frac{F(3 - 4\sin^2\gamma)}{4\sin(3\gamma)}$	$+\frac{F(4\cos^2\gamma - 1)}{4(2\cos^2\gamma - 1)\sin(3\gamma)}$	$-\frac{F(4\cos^2\gamma - 1)\times(4\cos^2\gamma - 3)}{8(2\cos^2\gamma - 1)\sin(3\gamma)}$

TABLE 9.15 Forces acting at the members of the radial reinforced trussed vault

Example

The forces are defined for a vault with $f = a/4$, radius $R = 2.125 \times a$, $\tan\beta = f/a$, so $\beta = 14.036^0$ and $\gamma = 7.018^0$. The structure is subjected to concentrated forces acting at the nodes: F = 1.0 kN. The following table shows the forces due to the symmetrical and asymmetrical loads acting at the nodes, for $F_i = 1.0$ kN.

Member	Sym load: F_2, F_4	Load at the top: F_3	Perm. load: F_2, F_3, F_4	Anti-metrical load: ½ $(-F_2, F_4)$	Asym load: ½(F_2,F_4)+ ½.$(-F_2, F_4)$
S_{34}	- 3.848	- 3.970	- 7.818	0	- 1.924
S_{36}	+0.940	- 0.030	+ 0.910	0	+ 0.470
S_{46}	0	+ 2.046	+ 2.046	- 0.992	- 0.992
S_{56}	+3.970	+ 1.924	+ 5.894	+0.992	+ 2.977
S_{45}	-4.092	- 2.046	- 6.138	- 1.055	- 3.101
S_{26}	0	+ 1.148	+ 1.148	+1.055	+ 1.055
S_{23}	- 3.848	- 3.970	- 7.818	0	+ 1.924
S_{16}	+3.970	+ 1.924	+ 5.894	- 0.992	+ 0.993
S_{12}	- 4.092	- 2.046	- 6.138	+1.055	- 0.990

TABLE 9.16 Forces acting in the members of the radial reinforced vault

A concentrated load acting at the crown causes a compressive force in the vertical web bar S_{36} = -0.03 kN, but the permanent load causes a tensile force, S_{36} = +0.910 kN, so this web bar is tensioned. Due to the asymmetrical concentrated load acting at node 4 member S_{46} is compressed, S_{46} = +0.992 but due to the permanent load this tie is tensioned, S_{46} = +2.046. So this tie is tensioned if the asymmetrical load is smaller than the permanent load.

For a vault with a = 7.2 m and f = 1.8 m the loads acting at the nodes are:

Permanent load: $F_i = q_g R \beta = 3.2 \times 15.3 \times 14.036 \times \pi/180 = 12$ kN

Live load: $F_3 = ½\, q_e R \sin\beta = ½ \times 5.0 \times 15.3 \times 0.47059 = 9.28$ kN
$F_4 = ½\, q_e R \sin\beta + (a - q_e R \sin\beta)/2 = 5.0 \times ½ \times 7.2 = 18.0$ kN

The normal forces acting at member S_{46} are:
permanent load: $S_{46} = 2.046 \times 12.0 = +24.55$ kN
asym. live load: $S_{46} = -0.992 \times 18.0 = -17.96$ kN,

The force due to the permanent load is larger than the force dueto the asymmetrical live load , thus the tie S_{46} is tensioned.

The analysis shows the effect of decreasing the lever arm to avoid compressive loads acting in the web bars. To prevent the web bars to be subjected by compressive normal forces the designer can decrease the lever arm or increase the dead load.

§ 9.11 Example vault strengthened with ties running from a centre point

Actually the strengthened vault is statically indeterminate. For the vault analysed earlier the bending moments and forces are calculated using a Finite-Element program for a permanent load q = 3.2 kN/m and variable load q = 5.0 kN/m. The cables can be dimensioned very slender. A disadvantage of this structural system is the increase of the bending moments. The distance Δ has to be chosen carefully so the ties are tensioned and the bending moments are at minimum. The following table show the results for $\Delta/f = 0.6/1.8 = {}^{1}/_{3}$.

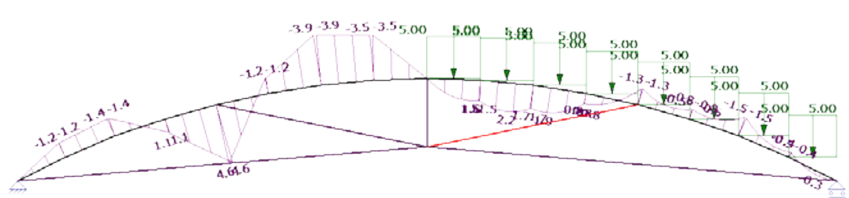

FIGURE 9.28 Bending moments due to the a-symmetrical live load

Node	x-coord.	y-coord.	Member	Dead load N	Dead Load M	Dead load V	Asym live load N	Asym live load M	Asym live load V
N1	-7.20	0.0	M1: n1-n2	- 64.1	0.87	2.28	- 27.8	1.22	1.31
N2	-6.360	-0.415	M2: n2-n3	- 62.8	1.94	2.51	- 27.8	1.40	0.19
N3	-5.497	-0.799	M3; n3-n4	- 61.5	1.94	3.83	- 27.7	1.40	2.67
N4	-4.612	-1.088	M4: n4-n5	- 60.5	2.15	3.41	- 27.5	4.61	3.77
N5	-3.711	-1.343	M5: n5-n6	- 71.9	2.15	4.09	- 53.9	4.61	6.18
N6	-2.795	-1.542	M6: n6-n7	- 71.4	1.48	2.73	- 54.2	3.90	2.90
N7	-1.869	-1.685	M7: n7-n8	- 70.9	1.36	1.62	- 54.3	3.90	0.42
N8	-0.937	-1.771	M8: n8-n9	- 70.6	1.36	2.95	- 54.1	3.50	3.73
N9	0	-1.80	M9: n9-n10	- 70.6	1.38	2.97	- 54.1	1.57	3.96
N10	0.937	-1.771	M10: n10-n11	- 71.0	1.52	1.54	- 54.4	2.18	2.57
N11	1.869	-1.685	M11: n11-n12	- 71.4	1.52	2.70	- 56.2	1.91	3.31
N12	2.795	-1.542	M12: n12-n13	- 71.9	2.07	4.07	- 56.2	1.27	4.43
N13	3.711	-1.343	M13: n13-n14	- 60.6	2.07	3.40	- 64.7	1.27	2.68
N14	4.612	-1.088	M14: n14-n15	- 61.6	1.59	3.39	- 66.1	1.46	2.79
N15	5.479	-0.799	M15: n15-n16	- 62.9	1.59	2.12	- 67.8	1.46	3.11
N16	6.360	-0.415	M16: n16-n21	- 64.2	0.90	2.31	- 69.7	0.41	2.32
N17	7.20	0	M17: n17-n18	+57.6			+62.0		
N18	0	-0.60	M18: n1-n18	+57.5			+25.6		
			M22: n9-n18	+ 4.3			+ 3.1		
			M20: n5-n18	+13.5			+29.1		
			M21: n13-n18	+13.4			- 8.0		

TABLE 9.17 Output for a surface load p_g = 3.2 kN/m and an asymmetrical load q_e = 5.0 kN/m

Due to the asymmetrical load the tie M21 is compressed, N = -9.0 kN, but due to the permanent loasd the tie is subjected to a tensile force N = + 13.4 kN, so the tie is tensioned permanently.i
Due to the permanent load q = 3.2 kN/m and variable load q = 5.0 kN/m the bending moment is at maximum for node 5.

N_{rep} = -71.9 – 53.9 = - 125.8 kN, σ = N/A = -2.52 MPa
M_{rep} = 2.15 + 4.61 = 6.76 kNm, σ = M z/I = 4.0 MPa

Thus the bending tensile stress is larger than the normal compressive stress.

Ultimate state

Due to the permanent load q = 3.2 kN/m and a variable load q = 5.0 kN/m acting at one half of the vault the bending moment is at maximum for node 5.

N_d = - 1.2 × 71.9 – 1.5 × 53.9 = - 167.1 kN, σ = N/A = -3.34 MPa
M_d = 1.2 × 2.15 + 1.5 × 4.61 = 9.5 kNm, σ = M.z/I = 5.68 MPa

For prefabricated structures the quality of the concrete is at least C35/45, with E_0 = 3.4 × 10^4 MPa. For the ultimate state the deformations increase by creep. For the design of the structure the stiffness of the vault is approached with: EI = $E_{0,t}$ I, with: $E_{0,t} = f_{cd}/(1.75 \times 10^{-3})$ and $f_{cd} = f_{ck}/1.5$

For C35/45: $E_{0,t} = (35/1.5)/(1.75 \times 10^{-3}) = 13.3 \times 10^3$ MPa.

The critical buckling force according to Euler is equal to: $N_{cr} = \pi^2$ EI

s_1^2

For this structure the length of the vault between the nodes is equal to: $s_1 = \frac{1}{2}\beta R = 3.75$ m

$$N_{cr} = \frac{\pi^2 \times 92.917 \times 10^6 \times 13.3 \times 10^3}{(3.75)^2 \times 10^6} = 867 \times 10^3 \text{ N}; \quad n_{cr} = N_{cr}/N_d = 867/167.1 = 5.2$$

The maximum stress follows from: $\sigma_d = \frac{N_d}{A} \pm \frac{M_d z}{I} \times \frac{n_{cr}}{(n_{cr} - 1)}$

$$\sigma_d = \frac{-167.1 \times 10^3}{50 \times 10^3} \pm \frac{9.5 \times 10^6 \times 55}{92.917 \times 10^6} \times \frac{5.2}{(5.2-1)} = -3.34 \pm 6.96 \text{ MPa}$$

Shear stresses

The vault has a thickness h = 110 mm. The radius of the infill tubes is r = 30 mm. The centre to centre distance c of the tubes is 100 mm. the thickness of the flange above or under the tubes is 25 mm.
For node 5: $V_d = 1.2 \times 4.09 + 1.5 \times 6.18 = 14.2$ kN

The shear stress follows from: $\tau_{dmean} = V_d/(b\,t)$ $\quad \tau_{dmean} = \frac{14.2 \times 10^3}{1\,000 \times 2 \times 25} = 0.284$ MPa

Between two tubes the shear force is: $H_d = M_d/(\frac{1}{2} z)$
with: $\frac{1}{2} z = (110-25)/2 = 42.5$ mm

Due to the shear force the upper and lower flange is subjected to bending moments. The bending moment M acting at the lower and upper flange is: M = V c/4. The bending moment M acting at the strut between two tubes is two times the bending moment acting on the upper and lower flange:

$$M_d = 2 \times (\tfrac{1}{4} V_d\, c) \quad \rightarrow \quad M_d = \tfrac{1}{2} \times V_d \times 90 = 45 \times V_d$$

The shear force acting at the strut between the tubes follows from:

$$H_d = M_d/(\tfrac{1}{2} z) \quad \rightarrow \quad H_d = 45 \times 14.2/42.5 = 15.0 \text{ kN}$$

The shear stress is: $\tau_{dmean} = H_d/(b\,t) = 15000/(1000 \times 30) = 0.5$ MPa

This stress is smaller than the maximum shear stress.

Tuning Δ

The distance Δ has to be chosen carefully, so the ties are tensioned and the bending moments are at minimum. The thrust tensions the ties connecting the supports with the centre. Due to the inclination of these ties the permanent load stretches all the ties. Due to the asymmetrical load a tie can be subjected to a compressive normal force. The distance Δ has to be large enough, so the tensile forces acting at the ties, due to the permanent load, are larger than the compressive forces due to the live load. However increasing the distance Δ will increase the tensile forces acting at the ties, due to the permanent load, but increase the bending moments as well. The maximum distance Δ follows from $\varepsilon < \gamma$ with $\tan \varepsilon = \Delta/a$. A strategy to tune the structure and chose Δ optimal is to start with $\Delta = a \tan \gamma$ and decrease the distance Δ till a tie is compressed due to the asymmetrical load.

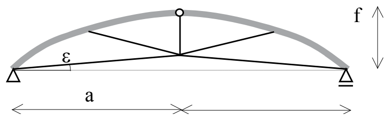

FIGURE 9.29 Prefabricated circular vault strengthened with ties running from a centre Caption

§ 9.12 Conclusions

A vault following a circle segment is structurally optimal for a radial load, for any other load the structure is subjected to bending moments. Strengthening these vaults is very efficient. Due to the strengthening the bending moments and shear forces decrease. In this chapter several options are described. Strengthening the vault with diagonals running from the crown to the supports reduces the bending moments significantly. However due to the asymmetric loads these diagonals are subjected to compressive as well as tensile normal forces, thus the vault has to be strengthened with tubes with a certain stiffness. For the given example the steel diagonals had to be dimensioned with a diameter of Ø100 mm to resist the compressive forces. The structure is very transparent in case the vault is strengthened radially with slender ties running from a centre to the vault. This centre is positioned above the supports at a distance Δ with $\Delta = a \times \tan(\varepsilon)$. The ties will be tensioned due to the inclination of the lower ties between the supports and centre. The distance Δ has to be chosen carefully, so the ties are tensioned continuously and the bending moments are at minimum. Of course strengthening vaults with radial ties is not exclusive for concrete shells. Arches of steel, timber or masonry can be strengthened in this way to decrease the need for material and the embodied energy in the structure.

10 Tests

Introduction

The previous chapters discussed the development of prefabricated vaults with tubular infill elements positioned perpendicular to the span of the vault. Due to the infill the stresses will increase just because the tubes reduce the area and the second moment of the area of the sections perpendicular to the span of the vault. The area is reduced proportionally with the ratio of the diameter of the tubes and height of the section. The tubes are positioned in the centre of the sections so the second moment of the area is reduced only slightly. Consequently the normal stresses will increase more than the bending stresses. Thus for vaults, subjected to compressive normal forces, the infill will increase the stiffness. However due to the infill the shear stresses will also increase. The shear stresses above and below the infill will increase proportionally with the ratio of the diameter of the infill elements and the height of the sections. The centre-to-centre distance of the infills will affect the shear stresses between the infills. The structure will collapse if the shear stresses surpass the ultimate shear stresses. Fortunately for vaults the bending moments and shear stresses are pretty small. This chapter discusses for a vault strengthened with diagonals and composed of prefabricated elements, two experiments to determine to define the structural resistance of these vaults with embedded cardboard tubes constructed perpendicular to the span.

§ 10.1 Description of the vault

The prefabricated vault is strengthened with two diagonals and composed of two segments. To facilitate the production the line of the system follows a segment of a circle. The rise of the vault is equal to f = 1.8 m and the span is equal to l = 2 a = 14.4 m. The self-weight and need of cement is reduced with an infill of cardboard tubes Ø60-1.4 mm. To analyse the effect of the centre-to-centre distance of the infill elements for the load bearing capacity two models are constructed with a varying distance. For test 1 and test 2 the centre-to-centre distance of the infill is respectively 100 mm and 90 mm.

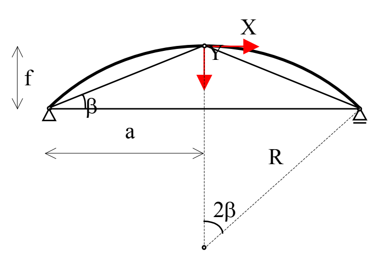

FIGURE 10.1 Prefabricated vault, the centre of the coordinates is positioned at the top

For the prefabricated vault the centre of the coordinates is positioned at the top. The coordinates follow from: x = R sin f and y = R (1- cos ϕ). The radius follows from expression [9.3]:

$$R = ½ (a^2 + f^2)/f$$

Substituting a = 7.2 m and f = 1.8 m gives R = 15.3 m. The angle β between the diagonal and horizontal line through the supports follows from $\tan \beta = a/f$. Substituting a = 7.2 m and f = 1.8 m givers $\tan \beta = 0.25$ and $\beta = 14.036^0$.

Prefabricated segments

The vault is composed of prefabricated segments. For these segments the centre of the coordinates is positioned at the top of the segment. The coordinates of the prefab segments follows from: $x = R \sin \phi$ and $y = R(1-\cos \phi)$, with R = 15.3 m. The span and rise are respectively equal to l_{seg} = 2 × 3.711 m and f_{seg} = 0.4568 m.

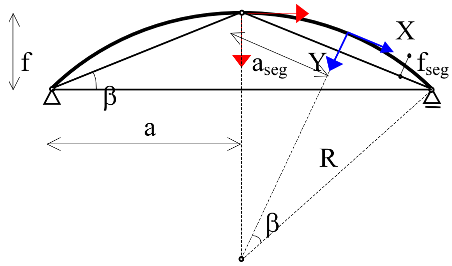

FIGURE 10.2 The centre of the coordinates of the prefabricated segment is positioned at the top of the segment.

Tested Elements

Two prefabricated elements are tested at scale 1:2. The radius R of the tested elements is equal to 7.65 m and the height, width and length of the sections of the elements are respectively h = 110 mm, b = 200 mm and l_{seg} = 2 × a_{seg} =2 × 1.856 m. For the tests the centre of centre distance of the cardboard tubes Ø60-1.4 varies. For test 1 and test 2 the centre-to-centre distance is respectively equal to c = 100 mm and c = 90 mm. Table 10.1 shows the coordinates. The angle β between the diagonal and horizontal line through the supports of the vault follows from $\tan \beta = a/f = 0.25$, so $\beta = 14.036^0$. The following table shows the coordinates.

angle ϕ	X =	y =
0	0	0
¼ β	0.464	0.014
½ β	0.928	0.057
¾ β	1.392	0.128
β	1.856	0.228

TABLE 10.1 Coordinates of the tested prefab elements. .

The area, second moment of the area, volume and self-weight are respectively:

$A = 200 \times (110 - 60) = 10^4\ mm^2$

$I = 200 \times (110^3/12 - 60^3/12) = 18.583 \times 10^6\ mm^4$

$V = 1000 \times 200 \times 110 - 10 \times 200 \times \pi \times 30^2 = 16.35 \times 10^6\ mm^3/m$

$q = 0.01635 \times 24 = 0.39\ kN/m$

The tested elements are subjected to the self-weight q = 0.39 kN/m and two concentrated loads ½ F acting at a distance c from the top.

Self weight

Due to the self-weight q = 0.39 kN/m the structure is subjected to a vertical reaction force equal to V = q R β. Substituting q, R and β gives: V = 0.73 kN.
The thrust H follows from expression [9.35'] .

$$H = \frac{\tfrac{1}{2} q R \times \{ \tfrac{7}{6} \sin^3(2\beta) + \beta \cos(2\beta)[1 - 2\sin^2(2\beta)] - \tfrac{1}{2}\sin(2\beta)\}}{\beta + 2\beta\cos^2(2\beta) - \tfrac{3}{2}\sin(2\beta)\cos(2\beta)}$$

In this expression the angle 2 β describes the angle from the top to the support. For the segment the angle between the top and support is equal to β. Thus for this segment we have to substitute $2\beta_{seg} = \beta$ into expression [9.35']:

$$H = \frac{\tfrac{1}{2} q R \times \{ \tfrac{7}{6} \sin^3\beta + \tfrac{1}{2}\beta \cos\beta[1 - 2\sin^2\beta] - \tfrac{1}{2}\sin\beta\}}{\tfrac{1}{2}\beta + \beta\cos^2\beta - \tfrac{3}{2}\sin\beta\cos\beta}$$

Substituting q, R and β gives: $H = 1.966 \times q R/2 = 2.93$ kN

At the top the bending moment $M_{f=0}$ follows from: $M_{\phi=0} = V a_{seg} - H f_{seg} - \tfrac{1}{2} q a_{seg}^2$

Substituting q, H, a_{seg} and f_{seg} gives:

$$M_{\phi=0} = 0.73 \times 1.856 - 2.93 \times 0.228 - 0.39 \times 1.856^2/2 = 0.015 \text{ kNm}$$

For x = ½ a: $M_{\phi=\beta/2} = M_{\phi=0} + H y - q a_{seg}^2/8$

$$M_{\phi=\beta/2} = 0.015 + 2.93 \times 0.057 - 0.39 \times 1.856^2/8 = 0.014 \text{ kNm}$$

The maximum bending stress follows from: $\sigma = \frac{M \times \tfrac{1}{2} h}{I}$ [10.1]

Substituting M, h and I gives: $\sigma = \frac{0.015 \times 10^6 \times 110/2}{18.583 \times 10^6} = 0.044$ MPa

As showed later the stresses due to the self-weight are much smaller than the stresses due to the concentrated loads acting at the crown of the prefabricated element.

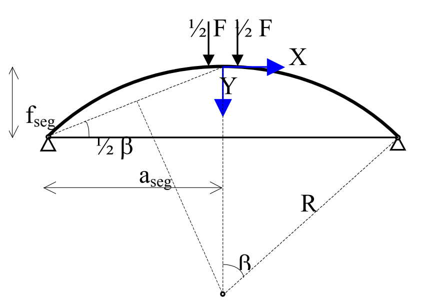

FIGURE 10.3 Segment subjected to the concentrated load acting at the crown

Concentrated load

The elements are subjected to two concentrated loads equal to ½ F acting at 0.09 m from the top. During the tests the forces are increased till the elements collapse. The reaction forces, thrust and bending moments are calculated for a force equal to F = 2 × 5.0 = 10 kN.
The reaction force is equal to V = 5.0 kN. The thrust H follows from expression [9.38']:

$$H = \frac{\tfrac{1}{2} F \times [\tfrac{1}{2} \sin^2\beta + \cos\beta\ (1 - \beta \sin\beta) - \cos^2\beta]}{{}^{1}/_{2}\,\beta + \beta \cos^2\beta - {}^{3}/_{2} \sin\beta \cos\beta} \qquad [9.38'']$$

Substituting q, R and β gives: $H = \tfrac{1}{2} F \times 6.326 = 31.63$ kN

At the top the bending moment $M_{\phi = 0}$ follows from:

$$M_{\phi = 0} = V\, a_{seg} - H\, f_{seg}$$

Substituting V, H, a_{seg} and f_{seg} gives: $M_{\phi = 0} = 5.0 \times 1.856 - 31.63 \times 0.228 = 2.07$ kNm

For x = ½ a_{seg} the bending moment $M_{\phi = \beta/2}$ follows from:

$$M_{\phi = \beta/2} = M_{\phi = 0} + H\, y - \tfrac{1}{2} F\, x$$

Substituting V, H, a_{seg} and f_{seg} gives: $M_{\phi = \beta/2} = 2.07 + 31.63 \times 0.057 - 5.0 \times 1.856/2 = 0.77$ kNm

The bending stress is calculated with expression [10.1]. Substituting the maximum bending moment M = 2.07 kNm, h = 110 mm and I into [10.1] gives:

$$\sigma = \frac{2.07 \times 10^6 \times 110/2}{18.583 \times 10^6} = 6.1 \text{ MPa}$$

The bending stress due to the concentrated load is much larger than the stress due to the self weight.

§ 10.2 Calculation with a finite element computer program

For the tested elements the bending moments and forces can be calculated more precisely, including the stiffness of the element and tie, with a computer program. The element is subjected to two concentrated loads ½ F = 5.0 kN acting at 0.09 m from the top. To minimize the deformation of the tie a very stiff tie was used ø48 mm, S235. The outward section of the concrete C28/35 is equal to 200×110 mm, thickness 25 mm. Table 10.2 shows the results.

At the top the bending moment and normal force are respectively M = 1.98 kNm and N = -28.73 kN. The shear force is equal to V = 3.93 kN. Due to the lengthening of the tie and the position of the two concentrated forces at a distance of 0.090 m from the top, the thrust and bending moment are slightly smaller than the thrust and bending moment defined previously for the statically indeterminate element, subjected to a concentrated force at the top. The normal forces increase from the top to the supports. At the support the normal force is at maximum: N = -30.0 kN. The bending moments and shear forces decrease from the top to the supports.

node	Coordinate x	Coordinate y	Member	Normal force	Moment	Shear force
1	0.090	0	M1: n1-n2	- 28.73	1.98	3.93
2	0.464	0.014	M2: n2-n3	- 28.90	0.58	2.11
3	0.928	0.057	M3: n3-n4	- 28.99	-0.87	2.34
4	1.392	0.128	M4: n4-n5	- 28.98	-0.34	1.13
5	1.856	0.228	M5: n1-n6	- 28.57	1.98	0
6	-0.090	0	M6: n6-n7	- 28.73	1.98	3.93
7	-0.464	0.014	M7: n7-n8	- 28.90	0.58	2.11
8	-0.928	0.057	M8: n8-n9	- 28.99	-0.87	2.34
9	-1.392	0.128	M9: n9-n10	- 28.98	-0.34	1.13
10	-1.856	0.228	M10: n5-n10	+28.57		

TABLE 10.2 Results from the analysis made with Matrix-frame

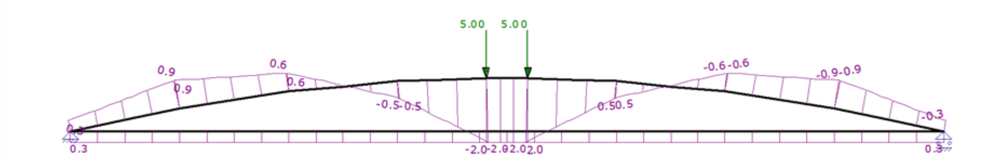

FIGURE 10.4 Bending moments due to concentrated loads

The stress due to the normal force and the bending moment follows from [10.2]. Substituting N = 28.73 kN, M = 1.98 kNm, h = 0.11 m, A = 10^4 mm^2 and I = 18.583 × 10^6 mm^4 into [10.2] gives:

$$\sigma = -\frac{N}{A} \pm \frac{M \times {}^1/_2\, h}{I} \quad \rightarrow \quad \sigma = -\frac{28730}{10^4} \pm \frac{1.98 \times 10^6 \times 110/2}{18.583 \times 10^6} = -2.87 \pm 5.86 \text{ MPa}$$

The compressive normal stress does not compensate the tensile bending stress, so possibly the element will be cracked due to this loading.

§ 10.3 Bearing resistance of the elements.

The bearing resistance of the elements is defined with a spreadsheet Excel, see the following graph, figure 10.5, for varying normal forces, concrete C30/37, rebars 2Ø6. FeB500, coverage c =15 mm, A_s = 4 × π/4 × 6^2 = 113 mm^2. The tested elements are subjected to two concentrated loads ½ F acting during a short time, thus the deformations are not increased by creep or shrinkage. For the tests the bearing capacity is defined for the representative values f_s = 500 MPa and f_c = 30 MPa, ε_c = 0.0035.

Figure 10.5, shows the bearing capacity of the tested element. Table 10.3 gives the ultimate bending moments for varying values of the representative normal force.This table is constructed with the following expressions for the equilibrium of the forces and bending moments.

The normal force follows from the equilibrium of forces:

$$N = N_c + N'_s - N_s \qquad [10.3]$$

The bending moment follows from:

$$M = N_c (½ h - 0.5 x_u) + (N'_s + N_s) \times (½ h - d) \qquad [10.4]$$

Due to the infill the concrete compressive zone x_u is limited to the flange, so $x_u \le c_F$.
The concrete normal force is equal to:

$$N_c = b\, x_u\, f_c \qquad [10.5]$$

According to the Euro code the stress-strain diagram of the concrete is linear with $\sigma_c = f_c$ for $\varepsilon_c \ge 0.2 \times 0.0035$. The height of the compressed zone is reduced with a factor β.
For $x < 1.25\, c_F$ the reduced height of the compressive zone x_u is equal to 0.8 x.
For $x \ge 1.25\, c_F$ the height of the compressive zone x_u is equal to c_F.

The height of the flange follows from: $c_F = (h - D)/2$. Substituting h = 110 mm and the diameter of the tubes , D = 60 mm, gives: $c_F = (110 - 60)/2 = 25$ mm.. The normal force acting in the rebars at the compressive zone follows from:

$$N'_s = ½ A_s \sigma_{cs} \qquad [10.6]$$

with: $\sigma_s = \varepsilon_c E_s (x - d)/x \le f_s$

The normal force acting in the rebars at the tensioned zone follows from:

$$N_s = ½ A_s \sigma_{cs} \qquad [10.7]$$

with: $\sigma_s = \varepsilon_c E_s (h - d - x)/x \le f_s$

$N/(b\, h\, f_{cd})$	$M/(b\, h^2\, f_{cd})$
-0.0456	0.0192
-0.0010	0.0384
0.0717	0.0654
0.1281	0.0853
0.1779	0.1014
0.2117	0.1114
0.2164	0.1130
0.2199	0.1141
0.2226	0.1151
0.2273	0.1149
0.2382	0.1125
0.2465	0.1102
0.2529	0.1080
0.2584	0.1061
0.2632	0.1045
0,2674	0.1031
0.2711	0.1019
0.2743	0.1008
0.2773	0.0998
0.2799	0.0989

TABLE 10.3 Bearing capacity of the sections of the tested element

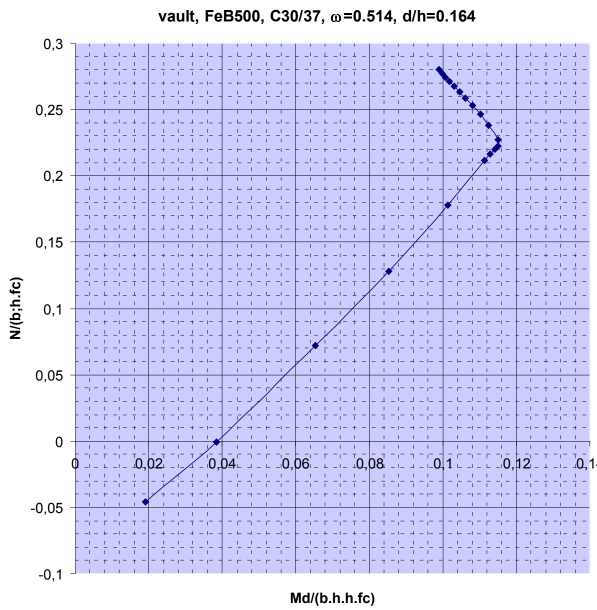

FIGURE 10.5 Graph showing the maximum bending moment with respext to the normal force for the tested elements

Due to the two concentrated loads acting at the top, equal to ½ F = 5.0 kN, the bending moment and normal force are respectively M = 1.98 kNm and N = 28.73 kN. For the normal force N = 28.73 kN the bending moment follows from table 10.3 with:

$$\frac{N}{b\,h\,f_c} = 0.044 \quad \rightarrow \quad \frac{M}{b\,h^2 f_c} = 0.055$$

Substituting A = 10^4 mm^2, f_c = 30 MPa and h = 110 mm gives M = 4.0 kNm

The graph shows that for $N/(b\,h\,f_c) < 0.2$ the maximum bending moment increases nearly linearly with an increasing normal force. For test 1 and test 2 the ultimate load at the element, following from the ultimate bending moment, is approximately equal to:

$$F_u = (2 \times 5.0) \times M_u/M \quad \rightarrow \quad F_u = 10.0 \times 4.0/1.98 = 20.0 \text{ kN}$$

Shear forces and stresses

For the two concentrated loads acting at the top, equal to ½ F = 5.0 kN, the maximal shear force acting at the vault just beside the top is equal to: V = 3.93 kN.The shear force acting at a section is equal to V = dM/dx. The shear force is distributed over the upper and lower flange with V_{up} and V_{low}. For a structure subjected to normal forces the shear forces in the upper and lower flange can be varying: $V_{up} \neq V_{low} \neq ½\, V_{lmean}$.
The mean shear stress acting at the flanges follow from:

$$\tau_{mean} = \frac{V}{2\,b\,x_u} = \frac{3930}{2 \times 200 \times 25} = 0.393 \text{ MPa}$$

According to the Eurocode the ultimate shear stress follows from:

$$\tau_{RDc} = \frac{V_{RDc}}{b\,h} = 0.12 \times k\,(100\,\rho\,f_{ck})^{1/3} + 0.15 \times \sigma_{\chi p} \geq 0.035 \times k^{1.5} \times \sqrt{f_{ck}} \text{ MPa} \qquad [10.8]$$

With: $k = 1 + \sqrt{(200/d)} \leq 2.0$, for $d = x_u = 25$ mm the factor k is equal to $k = 2.0$
The compressive strength is for C30/37 equal to $f_{ck} = 30$ MPa

$\rho = A_s/(b\,h) < 0.02 \quad \rightarrow \quad \rho = 113/(200 \times 50) = 0.0113$

$\sigma_{cp} = N/(b\,h) = 28730/(200 \times 50) = 2.87$ MPa

Substituting the parameters into [10.8] gives the ultimate shear stress:

$$\tau_{RDc} = 0.12 \times 2.0 \times (1.13 \times 30)^{1/3} + 0.15 \times 2.87 \geq 0.035 \times 2.0^{1.5} \times \sqrt{30} \text{ MPa}$$

$$\tau_{RDc} = +0.78 + 0.43 = 1.21 \geq 0.54 \text{ MPa}$$

For the flanges of test 1 and test 2 the shear resistance of the flanges follows from:

$$V_u = V \times \tau_u/\tau \quad \rightarrow \quad V_u = 3.93 \times 1.21/0.393 = 12.1 \text{ kN}$$

The shear resistance of the flanges is substantially, probably the shear forces acting at the flanges above and under the tubes are not decisive for the ultimate load.

For the struts between the tubes the horizontal shear force V_{strut} follows from the equilibrium of the bending moments:

$$V_{strut}\,(z_{up} + z_{low}) = V\,c \quad \rightarrow \quad V_{strut} = V\,c\,/(z_{up} + z_{low})$$

Substituting $z = z_{up} + z_{low}$ gives for the shear horizontal force acting at the strut:

$$V_{strut} = V\,c/z \qquad [10.9]$$

The mean stress acting at the section half way the height follows from:

$$\tau_{mean} = \frac{V\,c}{z\,b\,(c - D)} \qquad [10.10]$$

Test 1: for a centre-to-centre distance c = 100 mm, diameter of the tube: D = 60 mm, z = 110 - 25 = 85 mm and V = 3930 N the shear stress is equal to:

$$\tau_{mean} = \frac{V\,c}{z\,b\,(c-D)} = \frac{3930 \times 100}{85 \times 200 \times (100 - 60)} = 0.578 \text{ MPa}$$

Test 2: for a centre-to-centre distance c = 90 mm, D = 60 mm, z = 110 - 25 = 85 mm and V = 3930 N the shear stress follows from:

$$\tau_{mean} = \frac{V\,c}{z\,b\,(c-D)} = \frac{3930 \times 90}{85 \times 200 \times (90 - 60)} = 0.694 \text{ MPa}$$

According to the Euro-code the ultimate shear stress follows from [10.8]:

$$\tau_u \geq 0.035 \times k^{1.5} \times \sqrt{f_{ck}} \text{ MPa}$$

With: $k = 1 + \sqrt{(200/d)} \leq 2.0$, for $d = x_u = 25$ mm the factor k is equal to $k = 2.0$
The compressive strength is for C30/37 equal to $f_{ck} = 30$ MPa
The ultimate shear stress is at least equal to: $\tau_u = 0.035 \times 2.0^{1.5} \times \sqrt{30} = 0.54$ MPa

For test 1 and test 2 the ultimate load follows from: $V_u = V \times \tau_u/\tau$
Test 1: $V_u = 3.93 \times 0.54/0.578 = 3.67$ kN
Test 2: $V_u = 3.93 \times 0.54/0.694 = 3.06$ kN

The ultimate bending moment, the structure can resist, is much larger than the bending moments due to the load. For the flanges the ultimate shear force is much larger than the shear forces due to the load, but for the struts the ultimate shear forces are smaller than the shear forces due to the load. Consequently it is very likely that the models will not fail due to the bending moments or the shear forces acting on the flanges, but that for both elements the shear forces acting in the struts between the tubes will be critical. Decreasing the centre-to-centre distance of the infills will increase the shear stresses. Consequently the resitance of test 2 will be smaller than the resistance of test 1. Probably the ultimate shear stresses calculated with Eurocode are on the safe side, so possibly the tested elements can resist a larger load than following from the calculations.

§ 10.4 Validation with FEM program.

The distribution of the compressive, bending and shear stresses is analysed with three finite element models: a straight element loaded by a normal force, a straight element loaded by bending and a curved element loaded by bending.

Straight element subjected to normal load

To describe the distribution of the normal stresses a finite element model of a straight element was made. This model was subjected to two normal forces equal to N = 5.0 kN acting at both ends parallel to the span. The diameter of the infill is D = 60 mm, the height and width of the section are respectively h = 110 mm and b = 200 mm. The normal stresses acting on the flanges, above and below the infills, were approximately linear increasing from the upper and lower side to the infill. The mean stress acting in the flange is equal to:

$$\sigma_{mean} = \frac{\frac{1}{2} N}{b\, x_{flange}} \qquad [10.11]$$

Substituting, h = 110 mm, D = 60 mm, x_{flange} = ½ (h - D) = 25 mm, N = 5.0 kN and b = 200 mm gives:

$$\sigma_{mean} = \frac{\frac{1}{2} \times 5000}{200 \times 25} = 0.5 \text{ MPa}$$

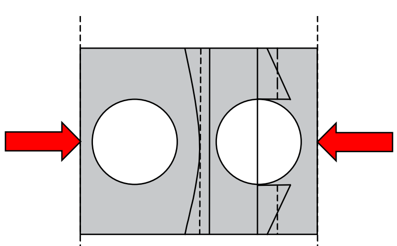

FIGURE 10.6 : Normal stresses due to a normal force N acting parallel to the span.

The minimal and maximal stress acting at the outer side and at the infill are equal to respectively: σ_1 = 0.28 MPa and σ_2 = 0.73 MPa:

The normal stresses follow from:

at the outer sides: $\sigma_1 = \beta\, \sigma_{mean} - \alpha\, \sigma_{mean}$
at the infill: $\sigma_2 = \beta\, \sigma_{mean} + \alpha\, \sigma_{mean}$

Substituting σ_1 = 0.28 MPa, σ_2 = 0.73 MPa and σ_{mean} = 0.5 MPa in these expressions gives:
at the outer sides: $\sigma_1 = \beta - \alpha = 0.28/0.5$
at the infill: $\sigma_2 = \beta + \alpha = 0.73/0.5$

Solving these expressions gives β = 1.01 and α = 0.45.
The normal stresses follow from:
at the outer sides: $\sigma_1 = 1.01 \times \sigma_{mean} - 0.45 \times \sigma_{mean} = 0.56 \times \sigma_{mean}$
at the infill: $\sigma_2 = 1.01 \times \sigma_{mean} + 0.45 \times \sigma_{mean} = 1.46 \times \sigma_{mean}$

So approximately for this structure with D/h = 60/110 the normal stresses acting on the flanges are at minimum and at maximum equal to:

$$\sigma = \sigma_{mean} \pm ½\, \sigma_{mean} \qquad [10.12]$$

Above and below the infill the normal stresses are at maximum at the infill: $\sigma = {}^3/_2\, \sigma_{mean}$

Shear stresses acting in the struts

The normal stresses between the holes were decreasing from the outer sides to the centre of section. The minimal and maximal stress acting at the outer side and at the centre are equal to respectively; σ_1 = 0.46 MPa and σ_2 = 0.11 MPa.

The mean stress is equal to: σ_{mean} = N/(b h), substituting N = 5.0 kN, h = 110 and b = 200 mm gives:

$$\sigma_{mean} = \frac{5000}{200 \times 110} = 0.23 \text{ MPa}$$

The normal stresses follow from:
at the outer sides: $\sigma_1 = \beta\, \sigma_{mean} + \alpha\, \sigma_{mean}$
at the centre: $\sigma_2 = \beta\, \sigma_{mean} - \alpha\, \sigma_{mean}$

Substituting σ_{mean} = 0.23 MPa, σ_1 = 0.46 MPa and σ_2 = 0.11 MPa gives:
at the outer sides: $\beta + \alpha = 0.46/0.23$
at the infill: $\beta - \alpha = 0.11/0.23$

Solving these expressions gives: β = 1.24 and α = 0.76. The normal stresses follow from:
at the outer sides: $\sigma_1 = 1.24 \times \sigma_{mean} + 0.76 \times \sigma_{mean} = 2.0 \times \sigma_{mean}$
at the infill: $\sigma_2 = 1.24 \times \sigma_{mean} - 0.76 \times \sigma_{mean} = 0.48 \times \sigma_{mean}$

So approximately for this structure with D/h = 60/110 the normal stresses are at minimum and at maximum equal to:

$$\sigma = {}^5/_4\, \sigma_{mean} \pm ¾\, \sigma_{mean} \qquad [10.13]$$

Between the infill the normal stresses are at maximum at the outher sides: $\sigma = 2\, \sigma_{mean}$

Straight element subjected to bending

To analyse the distribution of the bending stresses a second model was made. A straight element was subjected to two concentrated loads acting halfway the span with a centre to centre distance of c = 180 mm. Thus this straight element was subjected to bending and shear. The following figures show the results of the FEM calculation.

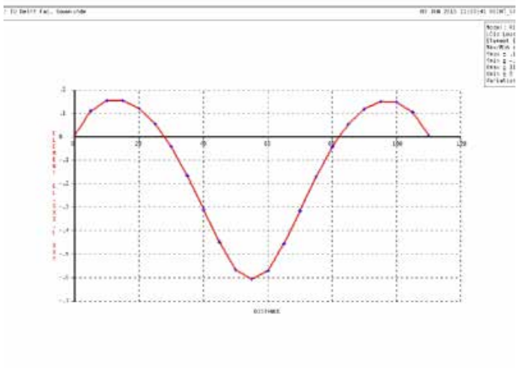

FIGURE 10.7 Distribution of the shear stresses in the struts between the infill

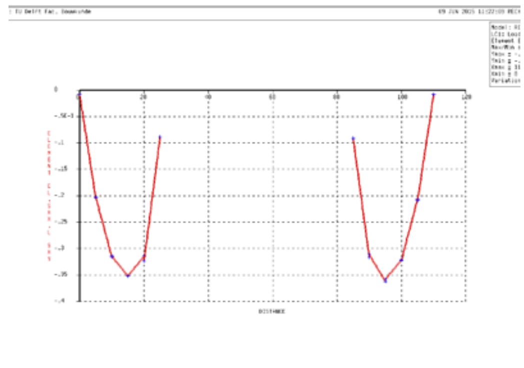

FIGURE 10.8 Distribution of the shear stresses in the struts above and below thw infill

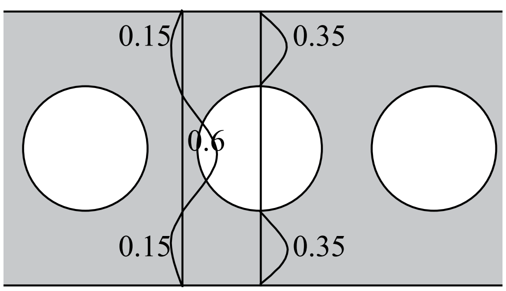

FIGURE 10.9 Shear stresses for a straight element with embedded infill subjected to bending due to the concentrated loads acting halfway the span.

Shear stresses acting at the flanges

According to this calculation the maximal shear stress acting at the flanges is equal to $\tau_{fl\,max}$ = 0.35 MPa. The sum of the shear stresses times the area must be equal to the shear force. For the zone above and below the infills the shear force follows from:

$$V = \beta\,\tau_{fl\,max}\,b\,(h - D)$$

Substituting V = ½ F = 2.5 kN, b = 200 mm, D = 60 mm and h = 110 mm gives:

$$2500 = \beta \times 0.35 \times 200 \times (110 - 60) \quad \rightarrow \quad \beta = 0.71$$

The mean stress is equal to: $\tau_{fl\,mean}$ = 0.71 × 0.35 = 0.25 MPa .The maximal shear stress is a factor 1/0.71 = 1.4 times the mean shear stress. For rectangular sections subjected to bending moments the shear stresses are distributed parabolically, then the maximum shear stress is 1.5 times the mean shear stress. The shear stresses above and between the infills are identical and approximately parabolic distributed over the flanges.

Shear stresses between the tubes

The shear stresses between the tubes were distributed as a curve with a maximum halfway the flanges and halfway the height. According to the FEM calculation the maximal shear stress acting in the flanges is equal to τ_{fl} = 0.15 MPa and the maximal shear stress acting halfway the height is equal to $\tau_{h/2}$ = 0.6 MPa. The sum of the shear stresses times the area must be equal to shear force. At the flanges of the section, at the upper and lower side, the shear stresses must compensate partly the shear force acting halfway the height. The following equation describes the equilibrium of shear forces:

$$\alpha\, \tau_{h/2}\, b\, D - \beta\, \tau_{fl}\, b\, (h - D) = V \quad \rightarrow \quad \alpha\, \tau_{h/2} + \beta\, \tau_{fl} - \beta\, \tau_{fl}\, (h/D) = V/(b\, D) \qquad [10.14]$$

For the flanges above and below the tubes the factor β is equal to 0.71. Substituting V =½ F = 2.5 kN, D = 60 mm, b = 200 mm, $t_{h/2}$ = 0.6 MPa , τ_{fl} = 0.15 MPa and β = 0.71 into expression [10.14] gives the factor α:

$$\alpha \times 0.6 + 0.71 \times 0.15 - 0.71 \times 0.15 \times 110/60 = 2500/(200 \times 60) \quad \rightarrow \quad \alpha = 0.495$$

Substituting α = 0.495, β = 0.71 and τ_{fl} = ¼ $\tau_{h/2}$ into [10.14] gives:

$$0.495 \times \tau_{h/2} + 0.71 \times ¼ \times \tau_{h/2} - 0.71 \times ¼ \times \tau_{h/2} \times (h/D) = V/(b\, D)$$

$$\tau_{h/2} = \frac{V/(b\, D)}{0.495 + 0.1775 \times (1 - h/D)} \qquad [10.14']$$

Substituting D = 60 mm, b = 200 mm, h = 110 mm into expression [10.14'] gives: $\tau_{h/2}$ =0.6 MPa

The shear stresses are affected by the bending moments acting at the struts between the tubes. Due to the infill the section is at minimum halfway the height between the infill, so the shear stress is at maximum. The mean stress acting at the section halfway the height follows from [10.10]:

$$\tau_{mean} = \frac{V\, c}{z\, b\, (c - D)} \qquad [10.10]$$

The element was subjected to two forces equal to ½ F = 2.5 kN. Substitute c = 90 mm, b = 200 mm, D = 60 mm and z = h − x_{flange} = 110 − 25 = 85 mm into the expression [10.10] to define the mean shear stress halfway the height of the section.:

$$\tau_{mean} = \frac{2500 \times 90}{200 \times 85 \times 30} = 0.44 \text{ MPa}$$

Comparing the maximum shear stress with the mean shear stress calculated with [10.10] shows that the maximum shear stress is about 0.6/0.44 =1.36 times the mean shear stress as calculated with [10.10]. For a rectangular section subjected to a bending moment the shear stress distribution follows a parabola, then the maximum stress is 1.5 times the mean stress. The ratio of the maximum and mean shear stress is smaller than 1.5. The schematization is on the safe side. The results, calculated with the FEM analysis, confirm the results calculated with the vault schematised as Vierendeel truss.

Curved element

To analyse the distribution of the stresses acting in the sections of a curved element a model was made with a curved element, subjected to two concentrated forces acting at 90 mm from the top. This curved element is subjected to the thrust and the concentrated load acting laterally, so the stresses are a combination of the stresses defined for the straight elements subjected to a normal force and the bending due to the concentrated loads acting laterally. Due to the thrust and the curvature of the element the shear stresses decrease from the centre to the supports.Figure 10.10 and 10.11 show the results of the FEM calculation.

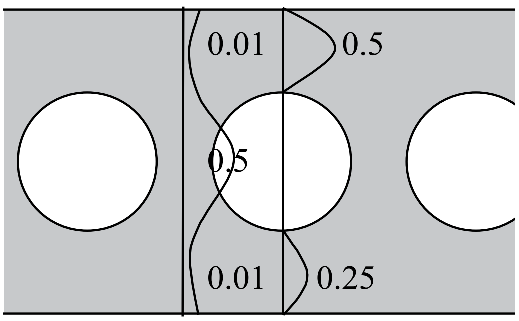

FIGURE 10.10 Shear stresses above and below the tubes and between the tubes

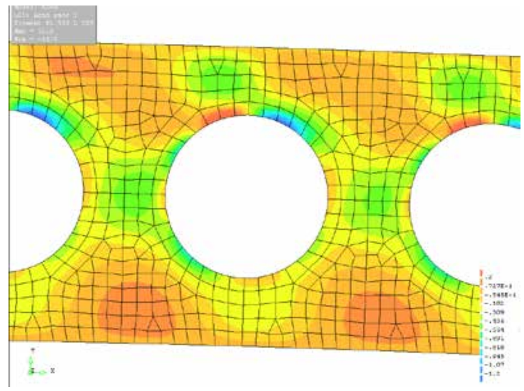

FIGURE 10.11 Distribution of the shear stresses for a vault with embedded infill subjected to bending due to two concentrated loads acting halfway the span.

Due to the normal forces acting at the sections the shear stresses acting in the upper and lower flange are varying: $V_{up} \neq V_{low} \neq ½\, V_{lmean}$. According to the FEM calculation the maximal shear stress acting in the upper flange is equa l to t_{fup} = 0.5 MPa, the maximal shear stress acting in the lower flange is equal to τ_{flow} = 0.25 MPa. Due to the distribution of the shear stresses the maximum stress is equal to the mean stress times a factor $1/\beta$. thus $\tau_{mean} = \beta\, \tau_{max}$. The sum of the shear stresses times the area must be equal to the shear force. For the flange above and below the tubes the shear force follows from:

$$V = \beta\tau_{up}\, b\,(h - D)/2 + \beta\, \tau_{low}\, b\,(h - D)/2$$

Substituting V = ½ F = 2.5 kN, b = 200 mm, D = 60 mm and h = 110 gives:

$$2500 = \beta \times (0.5 + 0.25) \times 200 \times (110 - 60)/2 \quad \rightarrow \quad \beta = 0.666$$

The maximal shear stress is equal to a factor 1/0.666 = 1.5 times the mean shear stress. The distribution of the shear stesses is parabolic. The mean shear stress is in the upper and lower flange respectively equal to: $\tau_{mean\ up}$ = 0.666 × 0.5 = 0.333 MPa and $\tau_{mean\ low}$ =0.666 × 0.25 = 0.167 MPa.

The normal force acting at the upper flange is greater than the normal force acting at the lower flange, consequently the shear resistance in the upper flange is greater than the shear resistance of the lower flange. To be on the safe side it is recommended for vaults to distribute the shear stress over the flanges proportional with the normal compressive force acting at the upper and lower flange.

For the vault the normal forces acting at the flanges follow from:

$$N_{fl} = -½\, N \pm M/z \qquad [10.15]$$

The analysis with Matrixframe showed that due to the concentrated load F = 10 kN the thrust is equal to 28.73 kN, the bending moment at the top is equal to M = 1.98 kNm and the shear force is equal to V = 3.93 kN. Substituting the bending moment, M = 1.98 kNm, the normal force N = 28.73 kN and z = 0.085 m gives:

$$N_{fl\ up} = -28.73 - 1.98/0.085 = -52.0 \text{ kN},$$

$$N_{fl\ low} = -28.73 + 1.98/0.085 = -5.4 \text{ kN},$$

The shear force acting at the upper flange is equal to: $V_{fl\ up} = \dfrac{3.93 \times 52}{52.0 + 5.4} = 3.56$ kN

The mean stress is equal to: $\tau_{mean} = \dfrac{V}{b \times (h - D)/2} = \dfrac{3560}{200 \times 25} = 0.71$ MPa

According to Euro-code the ultimate shear stress follows from:

$$\tau_{RDc} = \frac{V_{RDc}}{b\,h} = 0.12 \times k\,(100 \times \rho\, f_{ck})^{1/3} + 0.15 \times \sigma_{\chi p} \geq 0.035 \times k^{1.5} \times \sqrt{f_{ck}} \text{ MPa}$$ [illegible]

Substituting: $k = 2.0$; $f_{ck} = 30$ MPa; $\rho = A_s/(b\,h) = 0.0113$ and

$$\sigma_{cp} = N/(b\,h) = 52.0 \times 10^3/(200{\times}25) = 10.4 \text{ MPa}$$

$$\tau_{RDc} = 0.12 \times 2.0 \times (1.13{\times}30)^{1/3} + 0.15 \times 10.4 \geq 0.035 \times 2.0^{1.5} \times \sqrt{30} \text{ MPa}$$

$$\tau_{RDc} = +0.78 + 1.56 = 2.34 \text{ MPa} \geq 0.54 \text{ MPa}$$

For the upper flange the shear resistance increases due to the increase of the compressive normal force acting at the upper flange.

For the lower flange the compressive normal force is very small, the shear resistance is at least minimal: $\tau_{RDc} \geq 0.54$ MPa

The shear resistance for the upper flange follows from:	$V_u = 2.34 \times 200 \times 25 = 11.7 \times 10^3$ N
The shear resistance for the lower flange follows from:	$V_u = 0.54 \times 200 \times 25 = 3.2 \times 10^3$ N
The shear resistance of both flanges is equal to:	$V_u = 14.9 \times 10^3$ N

Thus the shear resistance is larger than the force V = 3.93 kN.

Shear resistance between the tubes

The shear stresses between the infills were distributed as a curve halfway the height and linear at the flanges. The mean stress acting at the section half way the height follows from [10.10]:

$$\tau_{mean} = \frac{V\,c}{z\,b\,(c - D)} \qquad [10.10]$$

The element was subjected to two forces equal to ½ F = 2.5 kN. Substitute c = 90 mm, b = 200 mm, D = 60 mm and $z = h - x_u = 110 - 25 = 85$ mm into expression [10.10] to define the mean shear stress:

$$\tau_{mean} = \frac{2500 \times 90}{85 \times 200 \times 30} = 0.44 \text{ MPa}$$

According to the FEM calculation the maximal shear stress acting in the flanges is equal to τ_{flange} = 0.01 MPa and the maximal shear stress acting halfway the height is equal to τ_{max} =0.5 MPa. So the maximal stress acting halfway the height between the infills is equal to 0.5/0.44 = 1.14 times the mean stress calculated with [10.10]. For the curved structure the shear stress 0.5 MPa is smaller than the shear stress defined for the straight element,.τ_{max} =0.6 MPa. Due to the curvature of the element the shear force is reduced due to the components of the vertical force and horizontal thrust acting perpendicular to the line of the system. For a rectangular section subjected to a bending moment the shear stress distribution follows a parabola, then the maximum stress is 1.5 times the mean stress. Probably the schematization as Vierendeel truss is on the safe side.

§ 10.5 Construction of the tested elements

For both tests a mould of multiplex was fabricated. Two rebars Ø6 FeB500 were positioned in the mould with a cover of 15 mm. The infills Ø60 were fixed to the sides of the mould to prevent these elements to be pushed upward due to the liquid concrete. Above the infills two rebars Ø6 FeB500 were laid and fixed to the infills with thin steel wire. The tubes were covered with a thin plastic foil so the unhardened concrete did not weaken these elements. To fill the spacing between the infills and mould the gravel was 8 mm at maximum. The consistence of the concrete, CEM 111/B43.5 N was rather stiff, class 3, to prevent the concrete from flowing downward to the ends.

The strength of the cubes was tested at 26-3-2015 and at 11-5-2015 with cubes 150×150×150 mm^3. Table 10.4 shows the results of the compressive tests.

	17-4-2015		11-5-2015	
Number	Strength [KN]	Compressive stress [MPA]	Strength [KN]	Compressive stress [MPA]
1	955	42.4	1051	46.7
2	895	39.7	1073	47.7
3	928	41.2	1017	45.2
4	947	42.1	1002	44.5
5	894	39.8	1069	47.5
6	916	40.7	1006	44.7
7			1050	46.7
8			1049	46.6
9			1081	48.1
Mean value:	923	41.0	1044	46.4

TABLE 10.4 Results of the compressive tests on cubes 150×150×150 mm^3.

The deviation s of a number of n tests follows from:

$$\sigma = [(x_i - x_{mean})^2 / n]^{1/2}$$

The ultimate compressive strength and the class of the concrete follows from:

$$f_{ck\ cube} = x_{mean} - 1.53 \times s$$

For n = 8 the deviation is equal to s = 1.245 MPa, Substituting s and x_{mean} gives the maximum compressive stress for the cubes:

$$f_{ck\ cube} = 46.4 - 1.53 \times 1.245 = 44.5 \text{ MPa}$$

According to this compressive strength the class is at least equal to C30/37.

§ 10.6 Results of test 1 and test 2

The elements test 1 and test 2 collapsed at a concentrated load of respectively F = 17.7 kN and 16.4 kN. Firstly the elements cracked at the lower side near the top due to the bending. Due to these small cracks the elements did not fail but then the elements were cracking between the infills. Finally the elements collapsed when the cracks were extended to the compressive zone. The sections of the elements were cracked totally with cracks running diagonally from the lower to the upper side.

To analyse the ultimate bearing capacity of the vault the results of the test are compared with the analysis. The analysis showed that due to the concentrated load F = 10 kN the thrust is equal to 28.73 kN, the bending moment at the top is equal to M = 2.0 kNm and the shear force is equal to V = 3.93 kN.The table and the graph, describing the ultimate resistance of the structure, show that for sections subjected to relative small normal forces the bending moment increases proportional with the normal force. For the ultimate loads found for test 1 and test 2 the ultimate bending moments and shear forces are calculated.

FIGURE 10.12 Tested element

Test 1

The element failed for a load F = 17.7 kN. For this load the thrust, the bending moment and shear force are respectively H = 50.9 kN, M = 3.5 kNm and V = 6.96 kN. The ultimate bending moment follows from table 10.3. The maximum bending moment increases with an increasing normal force. For N = 50.9 kN the maximum bending moment follows from table 10.3, with:

$$\frac{N}{b\,h\,f_c} = 0.077 \qquad \rightarrow \qquad \frac{M}{b\,h^2\,f_c} = 0.067$$

Substituting f_c = 30 MPa, b = 200 mm and h = 110 mm gives M_u = 4.9 kNm > M = 3.5 kNm

The ultimate bending moment is much larger than the calculated bending moment due to the maximum load, so the tested element did not fail due to the bending maximum moment but failed due to the shear forces. For the flanges the analysis showed an ultimate resistance equal to: V_u = 12.1 kN, this ultimate shear force is much larger than the shear forces acting at the flanges due to the load, so probably the shear force acting at the flanges is not critical.
For test 1 the calculated load the structure can resist, following from the ultimate shear force, is equal to: V_u = 3.67 kN. The vault can resist a shear force equal to V_u =6.96 kN, so the calculation is on the safe side.

FIGURE 10.13 Test 1, the structure fails just beside the top

FIGURE 10.14 Test 2, the structure fails just beside the top

Test 2:

The element failed for a load F_u = 16.4 kN, for this load the thrust, the bending moment and shear force are respectively H_u = 47.1 kN, M_u = 3.35 kNm and V_u =6.4 kN. The maximum bending moment follows from graph and table showing the ultimate bearing capacity. The ultimate bending moment increases with an increasing normal force. F or N_d = 47.1 kN the bending moment follows from table 10.3 with:

$$\frac{N_u}{b\,h\,f_c} = 0.07 \qquad \rightarrow \qquad \frac{M_u}{b\,h^2\,f_c} = 0.065$$

Substituting f_c = 30 MPa, b = 200 mm and h = 110 mm gives M_u = 4.7 kNm > 3.35 kNm

The ultimate bending moment is much larger than the calculated bending moment due to the maximum load, so the tested element did not fail due to the bending maximum moment but failed

due to the shear forces. For the flanges the analysis showed an ultimate resistance equal to: V_u = 12.1 kN, this ultimate shear force is much larger than the shear forces acting at the flanges due to the load, so probably the shear force acting at the flanges is not critical.
For test 2 the calculated load the structure can resist, following from the ultimate shear force, is equal to: V_u = 3.06 kN. The vault can resist a shear force equal to V_u =6.4 kN, so the calculation is on the safe side.

Conclusions

For both tests the maximum shear force due to the load than the calculated shear force. Probably the ultimate shear stresses calculated with the Eurocode are on the safe side. The ultimate shear stresses are calculated for concrete class C30/37. The mean compressive stress of the cubes was 46.4 MPa. Furthermore the rebars at the compressed side will increase the shear resistance, this will be discussed further in the following paragraph. The tests show that decreasing the centre-to-centre distance of the infills will decrease the resistance of the structure, this confirms the results calculated with the vault schematised as Vierendeel truss.

§ 10.7 Cracked vaults

The FEM analysis and the analysis for the vault schematised as Vierendeel-truss are based on the Theory of Elasticity. As shown by the experiments the vaults, subjected to normal forces, shear forces and bending moments, the cracking started at the tensioned side due to the bending moments. In a crack the tensile stresses are zero. Consequently, due to the cracking, the distribution of the forces and stresses changes. In a cracked section the shear stresses are transferred by the compressive zone, the reinforcement and the so-called interlock effect, caused by the friction acting in the cracks.
Generally for concrete beams the shear resistance is increased with stirrups. To define the shear resistance for these structures the beams are schematised as trusses composed of tensioned struts and compressive diagonals. In practice thin plates and vaults are reinforced with rebars only, mostly stirrups are not used. In the section cracked by bending the shear the compressive zone of the plate transfers the shear forces for the better part. For structures of concrete the shear resistance of the rebars is often neglected. Nevertheless the rebars , especially in the compressed zone, can contribute to the shear resistance of the structure.
A statically indeterminate structure fails in case more than one plastic hinge is formed. For statically indeterminate vaults the shear resistance of the rebars can help to redistribute the loads. Possibly the shear resistance of the rebars can prevent the structure to fall down suddenly.

For the rebars the ultimate shear stress τ_s follows from:

$$(\sigma_s^2 + 3\,\tau_s^2)^{0.5} < f_s \qquad [10.16]$$

For the rebars in the compressive zone the stress follows from the equilibrium of forces for the cracked section (10.3).

$$N = N'_c + N'_s - N_s \qquad [10.3]$$

For the tested vaults the shear resistance of the in the compressed zone is calculated. The compressive zone is equal to x. Due to the infills the concrete normal force is calculated in the flange with a height, c_F =½ × (110 -60) = 25 mm.The concrete normal force follows from:

$N'_c = \beta\, b\, x_u\, f_c$ [10.5]

for $x \le c_F / 0.8$ $\quad x_u = x$ and $\beta = 0.8$

for $x > c_F / 0.8$ $\quad x_u = c_F$ and $\beta = 1.0$

The normal force acting in the rebars at the compressive zone follows from:

$N'_s = \tfrac{1}{2} A_s \sigma_{cs}$ [10.6]

with: $\sigma_s = \dfrac{\varepsilon_c E_s (x - d)}{x} \le f_s$

The normal force acting in the rebars at the tensioned zone follows from:

$N_s = \tfrac{1}{2} A_s \sigma_{cs}$ [10.7]

with: $\sigma_s = \dfrac{\varepsilon_c E_s (h - d - x)}{x} \le f_s$

The normal load is small, so for this vault the stress in the tensioned rebar is at maximum. $\sigma_s = f_s$.
Substituting expression (10.5), (10.6) and (10.7) into (10.3) gives for $x \le c_F / 0.8$:

$0.8\, b\, x_u f_c + \tfrac{1}{2} A_s \varepsilon_c E_s (x - d)/x - \tfrac{1}{2} A_s f_s = N$ [10.17]

For test 1 the normal force is equal to N = 51.4 kN, substituting b = 200 mm, f_c = 30 MPa, f_s = 500 MPa , ½ A_s = 56 mm², ε_c = 0.0035, h = 110 mm and d = 18 mm in [10.17] gives:

$0.8 \times 200 \times 30\ x + 56 \times 0.0035 \times 2.0 \times 10^5 \times (x - 18)/x - 56 \times 500 = 51400$

$x = 20 \text{ mm} \le [c_F / 0.8 = 25/0.8 = 31.25 \text{ mm}]$

The stress acting in the rebars in the compressive zone follows from expression [10.6]:

$\sigma_s = \dfrac{\varepsilon_c E_s (x - d)}{x} \le f_s$

$\sigma_s = 0.0035 \times 2.0 \times 10^5 \times (20 - 18)/20 = 70 \text{ MPa} \le 500 \text{ MPa}$

For the rebars the ultimate shear stress t_s follows from [10.16]:

$(\sigma_s^2 + 3\,\tau_s^2)^{0.5} < f_s$ [10.16]

Substituting σ_s into [10.16] gives a maximum shear stress equal to: τ_s = 286 MPa

The rebars can resist a shear force equal to: $V_{us} = A_s \tau_s = 2 \times 28 \times 286 = 16 \times 10^3$ N.

For test 2 the normal force is equal to N = 46.3 kN, substituting b = 200 mm, f_c = 30 MPa, f_s = 500 MPa , ½ A_s = 56 mm², ε_c = 0.0035, h = 110 mm and d = 18 mm in [10.17] gives:

$0.8 \times 200 \times 30\ x + 56 \times 0.0035 \times 2.0 \times 10^5 \times (x - 18)/x - 56 \times 500 = 46300$

$x = 16 \text{ mm} \le [c_F / 0.8 = 25/0.8 = 31.25 \text{ mm}]$

The stress acting in the rebars in the compressive zone follows from expression [10.6]:

$\sigma_s = \dfrac{\varepsilon_c E_s (x - d)}{x} \le f_s$

$\sigma_s = 0.0035 \times 2.0 \times 10^5 \times (16 - 18)/16 = |-88| \text{ MPa} \le 500 \text{ MPa}$

For the rebars the ultimate shear stress t_s follows from [10.16]:

$(\sigma_s^2 + 3\,\tau_s^2)^{0.5} < f_s$ [10.16]

Substituting σ_s into [10.16] gives a maximum shear stress equal to: τ_s = 284 MPa

The rebars can resist a shear force equal to: $V_{us} = A_s \tau_s = 2 \times 28 \times 284 = 15.9 \times 10^3$ N.

For the tested vaults the rebars in the compressive zone can contribute substantially to the shear resistance of the structure. Further these bars will prevent the vault from failing suddenly if the structure is overloaded. Thanks to the contribution of the rebars in the compressive zone the tested elements can reist a larger load than predicted with the calculations according to the Eurocode.

§ 10.8 Conclusions

Probably the structure fails due to the shear stress acting between the infills. For vaults with embedded tubular infills the shear stresses acting in the struts between the infills can be approached with a scheme as Vierendeel-truss. The calculations with the FEM analysis show that the scheme as Vierendeel-truss predicts the stresses quite well. The tests show that the load bearing resistance of the structure is very good. Probably the schematization is on the safe side. Further research is needed to define the affect of the diameter and centre-to–centre distance of the infill elements for the shear resistance of the struts between the tubes.

11 Conclusions and recommendations

This research project focuses on the design of structures made in the past with an almost forgotten system and the possibilities of this system to design buildings nowadays and in the coming decades. To design buildings we need a lot of know-how. Buildings made in the past are a reservoir of knowledge and know-how, it would be a mistake if we neglect the lessons learned in the past. Studying this almost forgotten system can be helpfully to develop new systems capable to meet the demands of the present. The advantages and disadvantages of the Fusee Céramique system are analysed. Possible structural deficiencies are mentioned and possible solutions are described to overcome these deficiencies. The structural system is redesigned to meet the emerging problems building industry have to face nowadays. The warming up of the global temperature due to the greenhouse effect is a most important threat. To solve this problem many things have to be changed. Architects can contribute much to reduce the global warming up by designing buildings that use less energy for cooling, heating and construction. Especially roofs can contribute much to the realisation of environmental friendly buildings. For example green roofs can retain temporary rainwater and can contribute much to reduce the temperature variations and the energy needed to cool or heat buildings. Generally minimising the need of material for building construction will reduce the embodied energy too. For vaults the need of material is minimal. Due to the form the cylindrical vaults can transfer substantial loads without increasing the dimensions much. An infill can reduce the need of cement in floors, roofs and walls. In the past Jaques Couëlle developed the Fusee Céramique system to reduce the weight and need of cement for floors, walls and roofs of concrete. In those days the production and construction of this system was competitive. Unfortunately the construction of the vaults was very labour intensive. Fifty years ago the cost of labour was lower than the cost of materials. Nowadays the cost of labour is much higher than the cost of material. To be competitive the cost of labour must be reduced. To be cost effective the technique of construction has to be labouring extensive. Further an infill has to be sustainable. The CO_2 emission and the embodied energy of the production and construction must be minimal.
To show the competences of the Fusee Céramique system chapter 5 describes an example of a barrel vault. The cilindrical vault roofing a workshop, known as building Q, was very thin for a span of 19.8 m. With a thickness of 130 mm the ratio of the thickness versus the span was only $^{130}/_{19800} = {}^{1}/_{152}$. It is amazing to see how half a century ago engineers could design such slender vaults without help of computers. Can intuition and a real understanding of the load transfer compensate all the knowledge obtained in the last five decades? Actually knowledge can never compensate intuition, but of course intuition is fed by knowledge and experience. However analysing the vault of building Q in Woerden shows that this vault did not meet the demands of the present. Especially the effect of the time dependent deformations was underestimated.
Structurally structures composed of varying materials are complicated. Chapter 4 describes the decrease of the stiffness due to the time dependant effects. Embedding infill elements in a structure of concrete can change the load transfer if the shrinkage and creep of the infill elements and cement vary. For the Fusée Céramique tubes the shrinkage and creep is smaller than the shrinkage and creep of the cement, consequently the load transfer changes. The internal force acting at the fusées increase and the internal force acting at the concrete decrease. Generally a vault is subjected to a normal compressive load, but due to the time dependant affects the force acting at the fusées increases and the force acting at the concrete decreases so much that the concrete is subjected to a tensile force. Possible the structure cracks. Due to the cracks the stiffness will decrease substantially. For the Fusée Céramique vaults this effect is hazardously. Due to the decrease of the stiffness the buckling resistance will decrease too and possibly the structure falls down. The fact that fusée vaults could stand for fifty years or more proves only that the safety factor is larger than 1.0 , but does not prove that these

structures meet the safety demanded by the codes. Slender vaults constructed with one layer of fusées and a span of 14 m or more can be unsafe and have to be strengthened.
As shown in chapter 6 the existing vaults can be strengthened and stiffened easily with a simple truss composed of two diagonals. Due to the strengthening the bending moments decrease, the stiffness increases, the buckling length will decrease, consequently the buckling resistance will increase too.
To design these structures with a minimal need of material we have to understand very well the time dependent effects for structures composed of varying materials. If we compose structures of two or more materials with varying features then we have to be aware of the effect of these features on the load transfer.
Nowadays a designer has to choose an infill with respect to the structural features and the environmental impact. Chapter 8 describes the features of varying infill elements. A natural material as timber or bamboo or a composite of natural fibres as cardboard can be a good substitute for céramique elements. In the sixties of the past century the Fusee Céramique system was not competitive anymore. The construction of the roofs on the side, positioning the fusées one by one and pouring the liquid concrete in two layers was labour intensive. The low cost of construction could not compensate the reduction of the cost of cement and steel. Nowadays designers have to find a new cost effective method. For structures of concrete the cost of the moulds are substantial, reusing the moulds as much as possible reduces the cost significantly. Partitioning the roofs in small identical units and prefabricating these units in reusable moulds will reduce the cost much. The cost of the moulds can be reduced even more if the curvature is constant, consequently the line of system will be following a circle segment. Actually for cylindrical vaults a circle segment is structurally less efficient than a parabola. However due to the strengthening as described in chapter 9 the bending moments will be small and will not influence the stresses much.
Curving an infill according to the curvature of the vault is possible if the infill is not very stiff. Possible a supple infill can be composed of a bundle of small tubes of bamboo or straw. A stiff element has to be facetted to follow the curve of the vault. Otherwise a cylinder is curved in one direction. Changing the direction of the tubes offers the possibility to use straight elements. Chapter 10 described the construction of prefabricated elements with cardboard tubes perpendicular to the span.
For a structure subjected to a normal compressive force the normal stresses will increase reversibly with the ratio of the diameter of the tubes and height of the structure. For a structure of concrete the increase of the compressive normal stress is profitable. The increasing normal stress can compensate the tensile bending stress and prevent cracking of the structure. For low-rise vaults subjected to a substantial thrust it is favourable to position the infill tubes perpendicular to span provided the bending moments and shear stresses are small. Strengthening a vault with diagonals as described in chapter 6 and 9 reduces the bending moments and shear stresses and increases the buckling resistance substantially. A prefabricated strengthened vault with a thickness of 110 mm and a span of 14.4 m can transfer easily the heavy loads for a green roof usable for all kind public activities. A hollow core element with a span of 14.4 m can also transfer this load, but then the thickness is 400 mm at minimum. The thickness of the vault and the hollow core elements shows a ratio of 110/400 = 0.28 , so the need of material is much smaller.
For structures, subjected to bending, placing the tubular infill perpendicular to the span can increase the shear stresses. To define the shear stresses due to the infill the structure can be schematised as a Vierendeel-truss. The schematisation is tested with Finite Element calculations. Furthermore two prefabricated elements were constructed and put to the test. The experiments showed that the structures could transfer a larger loading than predicted with the calculations. The vaults with cardboard tubes placed perpendicular to the span can transfer the loads safely.

Conclusions

Due to the minimised need for materials form-active structures perform well and can be a good alternative to create environmental friendly buildings. However a form-active structure is structurally optimal if the line of thrust coincides the line of the system. For any other load the structure will be subjected to bending. In practice structures are subjected to varying loads, consequently these structures are subjected to bending moments. For the low rise vaults the differences between the curvature of a parabola, catenary or circle segment are quite small, generally smaller than the thickness of the vault, so the bending moments due to the loads are small too. Strengthening these structures with diagonals will be effective to reduce these bending moments.

Environmentally friendly infill elements will reduce the self-weight, embodied energy and emissions of buildings for roofs as well as floors.

Tubular infill elements can be positioned parallel or perpendicular to the span. For a structure of concrete, subjected to normal compressive force, positioning the infills perpendicular instead of parallel to the span will increase the normal stresses. Normal compressive stresses can compensate, at least partly, the tensile bending stresses and prevent the structure of cracking. Probably a vault with embedded tubular infills positioned perpendicular to the span is stiffer than a vault with tubular infills parallel to the span.

For a structure of concrete subjected to bending the positioning the infill elements perpendicular instead of parallel to the span will increase the bending stresses slightly but will increase the shear stresses above and below the infill elements proportionally with the reduction of the area. Further an increase of the width between the infill elements will decrease the shear stresses between the infill elements.

To define the shear and bending stresses the structure is schematised as a Vierendeel truss. This schematisation can be used for floors as well as vaults. For vaults the bending moments are much smaller than the bending moments acting on floors.Strengthening the vaults with diagonals will reduce these stresses further. Consequently for vaults the bending stresses and shear stresses are much lower than for floors. Probably the increase of the shear stresses due to the infill will not be decisive for the resistance of the strengthened structure.

Further research

To overcome the problems of in particular cities, owners must be stimulated to convert roofs to green and useful areas, to produce food or energy or for any other environmental friendly activities. Due to these activities the permanent and live loads will rise substantially. Classification of roofs for the these loads further, concerning a more or less limited accessibility for the public, will be helpful to design these roofs well or to strengthen existing roofs without increasing the cost of construction excessively. Due to the efficient load transfer form-active structures can transfer heavy loads for substantial spans without significantly increasing the need of material.

The possibilities to acquire technical knowledge from the past seem to be underestimated. Of course nowadays technical knowledge is much more than half a century ago, nevertheless we can learn much from the past and in particular from the buildings and structures, designed without the help of computers, advanced theories and extensive calculations . Mostly these buildings could transfer the loads safely for long periods. In some way these structures can be considered as long-term experiments; in this way the past can be a source of knowledge. Generally it is not allowed for public to be present when a building is pulled down. Nevertheless it is for researchers and designers most interesting to observe the demolishing of a building. Especially if it is possible to load the structure step by step till it falls down to define the ultimate bearing capacity.

To learn from the past the information must be accessible. Often drawings and calculations can't be found any more. Consultancies are ended or sold, for example if the owner is retired. Possibly archives are lost when the firm is sold or closed. By preference documents describing the design and all the calculations are collected, stored and archived by the government, digitally as well as in hard copy. Due to the fast progress in computer technology it can be difficult to read digital information stored in the past with hardware and software not available anymore.

This research explores the possibilities to use infill elements embedded in concrete vaults to reduce the embodied energy and CO_2 emissions. Possibly infill elements can be helpfully to reduce the environmental load of floors and walls too. Probably the described techniques to strengthen vaults and arches are useful too for arches and vaults of timber, steel, masonry, glass, bamboo, cardboard or any other material.

The resistance of the tested elements shows that the curved vault could transfer a larger load than predicted by the calculations. In this thesis the vaults with embedded infill are schemed as Vierendeel trusses with webbars and flanges. Probably this method to scheme these vaults is on the safe side. The tests gave an indication of the ultimate resistance of these structures with tubes perpendicular to the span. Probably an increase of the centre-to-centre of the infill elements near the supports will increase the resistance of the vault favourably. To define the shear stresses and ultimate shear resistance between the infill elements further, it is recommended to test more elements with a varying centre-to-centre distance and a varying diameter.

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Curriculum vitae

Martinus Willem (Wim) Kamerling was born 12 august 1950 in Amsterdam. From 1969 till 1977 he studied at the faculty of Architecture of the Technical university of Eindhoven. His master project was conducted within the department of Structural Design. From 1977 till 1984 he worked as structural engineer for D3BN consultancy, where he designed structures for varying buildings. For example: the plant WRK III in Enkhuizen; the concern of the Koninklijke Bijenkorf in Amsterdam; the office of De Nederlandsche Bank in Rotterdam and the structures for the offices and caissons for De Nederlandsche Bank in Alkmaar and Hoogeveen. From 1984 till 2000 he worked part-time at the faculty of architecture of Technische Hogeschool Utrecht were he teached structural design and mathematics. At the same time he was consultant and designed structures for varying projects, for example the renovation of the residential home Johanneshove in Huizen. From 1982 till 2015 he worked, firstly part-time and later full-time, as docent and researcher at the faculty of Architecture of the University of Delft, within the chair structural design. In 1989 he developed for students of architecture to design structures, a computer program Drako, based on the methods described by Livesly [Liv69], to define forces, moments and displacements in frames. He was co-author of varying books [Jon02], [Beu02], [Kam97] and papers for conference proceedings and magazines. Further he was a member of varying working groups , advising the director of education and the account manager of the library of the faculty of Architecture, the editorial staff of the journal Cement, as well as the board of the Betonvereniging, the ENCI Studiefonds and the Vereniging Bouwen met Staal. The research for this thesis was conducted from 2009 till 2015 within the chair of Structural Design. At the present he is retired and part-time structural adviser, reseacher and teacher.

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